MAT 225
Extra Credit - Exotic Vectors

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A vector space V is defined as

$$V = \{\vec{x} \mid -1 < x < 1\}$$

with the following operations on $(\vec{x}, \vec{y}) \in V$ and a scalar $c \in \mathbb{R}$, each of vector addition and scalar multiplication having both a hyperbolic trigonometric form and an algebraic form

$$\vec{x} \oplus \vec{y} = \tanh\left(\tanh^{-1}(x) + \tanh^{-1} y\right) \qquad c \odot \vec{x} = \tanh\left(c \cdot \tanh^{-1} x\right)$$

$$\vec{x} \oplus \vec{y} = \frac{x + y}{1 + xy} \qquad c \odot \vec{x} = \frac{(1+x)^c - (1-x)^c}{(1+x)^c + (1-x)^c}$$

We prove that V is a vector space by demonstrating the following properties of vectors using the operations above

1. V is closed under addition of vectors
2. Addition of vectors in V is commutative
3. Addition of vectors in V is associative
4. There is a vector additive identity $\vec{0} \in V$ such that $\vec{x} \oplus \vec{0} = \vec{x}$
5. For any $\vec{x} \in V$ there is a negation $-\vec{x} \in V$ such that $\vec{x} \oplus (-\vec{x}) = \vec{0}$
6. V is closed under multiplication by scalars in $\mathbb{R}$
7. Multiplication by scalars in distributes over addition of vectors in V
8. Multiplication by vectors in V distributes over addition of scalars
9. Multiplication of vectors in V by scalars is associative
10. There is a scalar multiplicative identity $s \in \mathbb{R}$ such that $s \odot \vec{x} = \vec{x}$

1 V is closed under addition of vectors

if $\vec{x} \oplus \vec{y} = \vec{z}$ and $\vec{z} \in V$

By definition $\vec{z} \in V$ if $-1 < z < 1$; in other words $|z| < 1$. Using the definition of $\vec{x} \oplus \vec{y}$,

$$|z| = \left| \frac{x+y}{1+xy} \right| < 1$$

We regroup by properties of absolute values and rearrange the inequality to isolate variables. We let $X = |x|$ and $Y = |y|$ for clarity:

$$\frac{|x+y|}{|1+xy|} \leq \frac{|x|+|y|}{1+|x|\,|y|} = \frac{X+Y}{1+XY} < 1$$

$$\begin{aligned} X + Y &< 1 + XY \\ X + Y - XY &< 1 \end{aligned}$$

factoring, we have

$$\begin{aligned} X + Y\,(1-X) &< 1 \\ Y\,(1-X) &< 1 - X \end{aligned}$$

which is falsified if and only if $Y \geq 1$; by definition $-1 < y < 1$ so $Y = |y| < 1$ and the inequality holds for any $\vec{x}$. Factoring $|x|$ gives an identical inequality that holds for any $\vec{y}$

$$\begin{aligned} Y + X(1-Y) &< 1 \\ X(1-Y) &< 1 - Y \end{aligned}$$

So $-1 < z < 1$ when $\vec{z} = \vec{x} \oplus \vec{y}$ therefore $\vec{x} \oplus \vec{y} \in V$ and V is closed under vector addition.

2 Addition of vectors in V is commutative

if $\vec{x} \oplus \vec{y} = \vec{y} \oplus \vec{x}$

First we write the definition of $\vec{x} \oplus \vec{y}$ substituting S and P for the sum and product terms, respectively

$$\vec{x} \oplus \vec{y} = \frac{x+y}{1+xy} = \frac{S}{1+P}$$

then we write the commuted operation in the substituted form

$$\vec{y} \oplus \vec{x} = \frac{y+x}{1+yx} = \frac{S}{1+P}$$

and observe the commutivity of real numbers

$$x + y = S = y + x$$
$$xy = P = yx$$
$$\vec{x} \oplus \vec{y} = \frac{S}{1+P} = \vec{y} \oplus \vec{x}$$

therefore vector addition in V is commutative.

3 Addition of vectors in V is associative

if $(\vec{x} \oplus \vec{y}) \oplus \vec{z} = \vec{x} \oplus (\vec{y} \oplus \vec{z})$

Here we use the hyperbolic tangent definition of $\vec{x} \oplus \vec{y}$

$$\vec{x} \oplus \vec{y} = \tanh\left(\tanh^{-1}(x) + \tanh^{-1} y\right)$$

We substitute A and B for the associative groups

$$(\vec{x} \oplus \vec{y}) \oplus \vec{z} = \vec{x} \oplus (\vec{y} \oplus \vec{z})$$
$$A \oplus \vec{z} = \vec{x} \oplus B$$

and write A and B in terms of the definition; then take their inverse hyperbolic tangents for substitution in the next step

$$A = \vec{x} \oplus \vec{y} = \tanh\left(\tanh^{-1}(x) + \tanh^{-1} y\right)$$
$$B = \vec{y} \oplus \vec{z} = \tanh\left(\tanh^{-1}(y) + \tanh^{-1} z\right)$$

$$\tanh^{-1} A = \tanh^{-1}(x) + \tanh^{-1} y$$
$$\tanh^{-1} B = \tanh^{-1}(y) + \tanh^{-1} z$$

Finally we write $A \oplus \vec{z} = \vec{x} \oplus B$ as $C = D$ where

$$C = \tanh\left(\tanh^{-1}(A) + \tanh^{-1} z\right) = \tanh\left(\tanh^{-1}(x) + \tanh^{-1}(y) + \tanh^{-1} z\right) = \tanh(S)$$
$$D = \tanh\left(\tanh^{-1}(x) + \tanh^{-1} B\right) = \tanh\left(\tanh^{-1}(x) + \tanh^{-1}(y) + \tanh^{-1} z\right) = \tanh(S)$$
$$\tanh(S) = \tanh(S)$$

so vector addition is associative in V.

4 There is a vector additive identity $\vec{0} \in V$ such that $\vec{x} \oplus \vec{0} = \vec{x}$

We define the zero vector $\vec{p}$ in terms of addition in V:

$$\begin{aligned}
\vec{x} \oplus \vec{p} = \frac{x+p}{1+xp} &= x \\
x + p &= x + x^2 p \\
x - x &= x^2 p - p \\
x - x &= \left(x^2 - 1\right) p \\
\frac{x-x}{x^2-1} &= p \\
0 &= p
\end{aligned}$$

By definition $-1 < x < 1$ so $p = 0$ is defined for all $\vec{x}$. Finally $-1 < 0 < 1$ so $\vec{p} = \vec{0} \in V$.

5 For any $\vec{x} \in V$ there is a negation $-\vec{x} \in V$ such that $\vec{x} \oplus (-\vec{x}) = \vec{0}$

Let $\vec{p} = -\vec{x}$ and define $\vec{p}$ in terms of addition in V:

$$\begin{aligned}
\vec{x} \oplus \vec{p} = \frac{x+p}{1+xp} &= 0 \\
x + p &= 0 \\
p &= -x
\end{aligned}$$

and if $\vec{x} \in V \Rightarrow -1 < x < 1$ then $|-x| = x < 1$ so $-x$ is in V.

6 V is closed under multiplication by scalars in $\mathbb{R}$

We use the hyperbolic tangent definition of $c \oplus \vec{v}$

$$c \oplus \vec{v} = \tanh\left(c \cdot \tanh^{-1} v\right)$$

We only need to observe that $-1 < \tanh x < 1$ for all x; if $c \oplus \vec{v} = \vec{u}$ then $-1 < u < 1$ for any $c \in \mathbb{R}$, $\vec{u}$ is in V, and V is closed under scalar multiplication.

$$\tanh x = \frac{e^x - {}^1\!/e^x}{e^x + {}^1\!/e^x} < 1$$

$$\begin{aligned}
e^x - {}^1\!/e^x &< e^x + {}^1\!/e^x \\
e^x &< e^x + {}^2\!/e^x \\
\tanh x &< 1 \text{ for all x}
\end{aligned}$$

$$\tanh x = \frac{e^x - {}^1\!/e^x}{e^x + {}^1\!/e^x} > -1$$

$$\begin{aligned}
e^x - {}^1\!/e^x &> -e^x - {}^1\!/e^x \\
e^x &> -e^x \\
\tanh x &> -1 \text{ for all x}
\end{aligned}$$

7 Multiplication by scalars distributes over addition of vectors in V

We again use the hyperbolic tangent definition of $c \odot \vec{x}$ as well as of $\vec{x} \oplus \vec{y}$

$$c \odot \vec{x} = \tanh\left(c \cdot \tanh^{-1} x\right)$$
$$\vec{x} \oplus \vec{y} = \tanh\left(\tanh^{-1}(x) + \tanh^{-1} y\right)$$

We will show that $c \odot (\vec{x} \oplus \vec{y}) = (c \odot \vec{x}) \oplus (c \odot \vec{y})$. Begin by grouping the products as X and Y and taking their inverse hyperbolic tangents

$$X = c \odot \vec{x} = \tanh\left(c \cdot \tanh^{-1} x\right)$$
$$Y = c \odot \vec{y} = \tanh\left(c \cdot \tanh^{-1} y\right)$$

$$\tanh^{-1} X = c \cdot \tanh^{-1} x$$
$$\tanh^{-1} Y = c \cdot \tanh^{-1} y$$

Evaluate the sum of products

$$\begin{aligned} X \oplus Y &= \tanh\left(\tanh^{-1}(X) + \tanh^{-1} Y\right) \\ &= \tanh\left(c \cdot \tanh^{-1}(x) + c \cdot \tanh^{-1} y\right) \\ (c \odot \vec{x}) \oplus (c \odot \vec{y}) &= \tanh\left[c \cdot \left(\tanh^{-1}(x) + \tanh^{-1} y\right)\right] \end{aligned}$$

and compare with $c \odot (\vec{x} \oplus \vec{y})$

$$\begin{aligned} c \odot (\vec{x} \oplus \vec{y}) = c \odot S &= \tanh\left(c \cdot \tanh^{-1} S\right) \\ S = \vec{x} \oplus \vec{y} &= \tanh\left(\tanh^{-1}(x) + \tanh^{-1} y\right) \\ \tanh^{-1} S &= \tanh^{-1}(x) + \tanh^{-1} y \\ c \odot S &= \tanh\left(c \cdot \tanh^{-1} S\right) \\ &= \tanh\left[c \cdot \left(\tanh^{-1}(x) + \tanh^{-1} y\right)\right] \end{aligned}$$

thus

$$c \odot (\vec{x} \oplus \vec{y}) = \tanh\left[c \cdot \left(\tanh^{-1}(x) + \tanh^{-1} y\right)\right] = (c \odot \vec{x}) \oplus (c \odot \vec{y})$$

$$c \odot (\vec{x} \oplus \vec{y}) = (c \odot \vec{x}) \oplus (c \odot \vec{y})$$

Scalar multiplication distributes over vector addition in V.

8 Multiplication by vectors in V distributes over addition of scalars

This proof is very similar to the previous one. Again using the definitions as

$$\begin{aligned} c \odot \vec{x} &= \tanh\left(c \cdot \tanh^{-1} x\right) \\ \vec{x} \oplus \vec{y} &= \tanh\left(\tanh^{-1}(x) + \tanh^{-1} y\right) \end{aligned}$$

Assume that c and d are real valued scalars; we then show that

$$(c+d) \odot \vec{x} = (c \odot \vec{x}) \oplus (d \odot \vec{x})$$

let

$$\begin{aligned} C = c \odot \vec{x} &= \tanh\left(c \cdot \tanh^{-1} x\right) \implies \tanh^{-1} C = c \cdot \tanh^{-1} x \\ D = d \odot \vec{x} &= \tanh\left(d \cdot \tanh^{-1} x\right) \implies \tanh^{-1} D = d \cdot \tanh^{-1} x \end{aligned}$$

then

$$\begin{aligned} (c \odot \vec{x}) \oplus (d \odot \vec{x}) &= C \oplus D \\ &= \tanh\left(\tanh^{-1}(C) + \tanh^{-1} D\right) \\ &= \tanh\left(c \cdot \tanh^{-1}(x) + d \cdot \tanh^{-1} x\right) \\ &= \tanh\left[(c+d) \cdot \tanh^{-1} x\right] \\ &= (c+d) \odot \vec{x} \end{aligned}$$

so vector multiplication in V distributes over scalar addition.

9 Multiplication of vectors in V by scalars is associative

With the definition of scalar multiplication

$$c \odot \vec{x} = \tanh\left(c \cdot \tanh^{-1} x\right)$$

show that $(cd) \odot \vec{x} = c \odot (d \odot \vec{x})$.

Let $D = d \odot \vec{x}$

$$D = \tanh\left(d \cdot \tanh^{-1} x\right) \implies \tanh^{-1} D = d \cdot \tanh^{-1} x$$

and compose multiplication

$$\begin{aligned} c \odot (d \odot \vec{x}) = c \odot D &= \tanh\left(c \cdot \tanh^{-1} D\right) \\ &= \tanh\left[c \cdot \left(d \cdot \tanh^{-1} x\right)\right] \\ &= \tanh\left(cd \cdot \tanh^{-1} x\right) \\ &= (cd) \odot \vec{x} \end{aligned}$$

and scalar multiplication of vector in V is associative.

10 There is a scalar multiplicative identity $s \in \mathbb{R}$ such that $s \odot \vec{x} = \vec{x}$

Here we use for the first time the algebraic definition of scalar multiplication in V:

$$c \odot \vec{x} = \frac{(1+x)^c - (1-x)^c}{(1+x)^c + (1-x)^c}$$

Assume the scalar multiplicative identity s is 1 so that $1 \odot \vec{x} = \vec{x}$. Substituting in the definition we have

$$1 \odot \vec{x} = \frac{(1+x)^1 - (1-x)^1}{(1+x)^1 + (1-x)^1} = \vec{x}$$

$$\begin{aligned}
(1+x) - (1-x) &= x\left[(1+x) + (1-x)\right] \\
1 + x - 1 + x &= x + x^2 + x - x^2 \\
1 - 1 + x + x &= x + x + x^2 - x^2 \\
2x &= 2x \\
1 \odot \vec{x} &= \vec{x}
\end{aligned}$$

V is a vector space.
