MAT 225 — Linear Algebra
Homework and Skills Overview

Avilez, Fall 2019
Linear Algebra and its Applications, 5th Edition, David Lay

How to use this outline

The homework is divided into chapters and sections according to the book. Each section is further divided into groups of exercises related by their focus on a particular skill or method.

Beside each group of exercises is a bar of three blocks that indicate where attention should be focused. In order from most to least important, these indications are

- never attempted;
- not adequately prepared;
- need practice;
- generally comfortable, but specific questions need practice; and
- comfortable.

Homework pages are numbered according to the chapter (Ch), section (§), subsection (§§), and page of work (p) in that subsection, written as

$$Ch.\S.\S\S - p\,.$$

Contents

Chapter 4

Vector Spaces

4.1 Vector Spaces and Subspaces

Exercises to practice

1–31, 35, 36

☐☐☐ **4.1.1 (1–4) Understand properties of vectors and vector spaces**

☐☐☐ **4.1.2 (5–8) Use properties of vectors to identify vector spaces**

Determine if the given set is a subspace of $\mathbb{P}_n$ for an appropriate value of n

5. All polynomials of the form $p(t) = at^2$ where $a \in \mathbb{R}$

Yes; the set is Span $\{t^2\}$, so $p(t)$ is a subset of at least $\mathbb{P}_3$ (which has the basis $[1, t, t^2]$).

☐☐☐ **4.1.3 (9–12) Use properties of subspaces to construct a vector that spans a specific line in $\mathbb{R}^3$**

9. Let H be the set of all vectors of the form $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix}$. Find a vector $\vec{v}$ in $\mathbb{R}^3$ such that $H = \text{Span}\{\vec{v}\}$. Why does this show that H is a subspace of $\mathbb{R}^3$?

Questions to ask

$H = \text{Span}\{\vec{v}\}$ is a subspace of $\mathbb{R}^3$ when $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ because Theorem $\boxed{4.1.1}$ in Lay (p. 196) states that the span of any vectors in a vector space V is a subspace of V.

☐☐☐ **4.1.4 (13–14) Understand the relationship between the span of a set of vectors and subspaces**

13. Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

(a) Is $\vec{w}$ in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$? How many vectors are in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

$\vec{w}$ is not in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$; there are three vectors in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and none of them is equal to $\vec{w}$.

(b) How many vectors are in Span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

There are infinitely many vectors in the span of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$: all the vectors that can be made by a linear combination of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ with weights on each. $\vec{v}_2$ and $\vec{v}_3$ are not independent, however, so they span the same vectors; that is Span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} =$ Span $\{\vec{v}_1\} \cup$ Span $\{\vec{v}_2\}$.

(c) Is $\vec{w}$ in the subspace spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$? Why?

We're asked if there is a linear combination of vectors in $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ that equals $\vec{w}$; i.e. is there any $\vec{c}_3$ such that $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{w}$?

At a glance we can see that $\vec{w}$ is the sum of $\vec{v}_1$ and $\vec{v}_2$ with weights $\langle 1, 1 \rangle$. In fact $\vec{v}_3$ and $\vec{v}_2$ are not independent so either, but not both, is necessary (the weights differ between choices):

$$[\vec{v}_1 \quad \vec{v}_3] \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} = [\vec{v}_1 \quad \vec{v}_2] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 1 \cdot 2 \\ 1 \cdot 0 + 1 \cdot 1 \\ 1 \cdot -1 + 1 \cdot 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

☐☐☐ 4.1.5 (15–18) Use properties of vector spaces to construct vectors that span a space W or to show why W is not a vector space

In exercises 15–18, let W be the set of all vectors of the form shown, where a, b, and c represent arbitrary real numbers. In each case, either find a set S of vectors that spans W or give an example to show that W is not a vector space.

15. $\begin{bmatrix} 3a + b \\ 4 \\ a - 5b \end{bmatrix}$

Notice that the second row does not depend on any variable and is not equal to zero; this means that any vector we construct in W will have a value of 4 in the second row, so we cannot construct the zero vector in W and W is not a vector space. For example let $a = b = 0$, and

$$\begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} \neq \vec{0}$$

Additionally if we added any two vectors in W then the resultant vector would not be in W and W would not be closed under addition (again not a vector space)

$$\begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 3a+b \\ 4 \\ a-5b \end{bmatrix} \text{ for any } (a,b)$$

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16. $\begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$

Represent W in parametric vector form:

$$\begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix} = a\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ -6 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Notice that there is a constant displacement $\langle 1, 0, 0 \rangle$ from the origin. The only possible zero vector then would be one with $a = 1$ to cancel that displacement in the first row; however when $a = 1$ then b must be positive to subtract to zero in the second row (specifically $b = 1/6$). When $a = 1$ and b is positive the third row will only have a positive nonzero value, so the zero vector is not in W and W is not a vector space.

Furthermore this displacement will cause addition in W to produce vectors outside of W, so W is not a vector space. For example,

$$\vec{u} = \boldsymbol{A}\begin{bmatrix} a & b \end{bmatrix} + \vec{c} = \begin{bmatrix} -1 & 0 \\ 1 & -6 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{u} + \vec{u} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{A}\begin{bmatrix} a & b \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{A}\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

If $\vec{u} + \vec{u}$ is in W then there must be some solution for $\langle a, b \rangle$ that satisfies this equation. We augment $\boldsymbol{A}$ and put the augmented matrix in reduced echelon form

$$\left[\begin{array}{cc|c} -1 & 0 & 1 \\ 1 & -6 & 0 \\ 1 & 2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

and we can see from the third row that there is no solution $(0 \neq 1)$.

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4.1.6 (19) Applications: use properties of vector spaces to show that simple harmonic oscillators comprise a vector space

19. A mass m is fixed to the end of a spring; the mass is pulled downward and released so that the mass and spring oscillate in vertical motion. The displacement y of the mass from its resting position is given by a function of the form

$$y(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

where ω is a constant specific to each spring–mass combination. Show that the set of all functions described by this form (with ω fixed and c_1, c_2 arbitrary) is a vector space.

Both $\cos \omega t$ and $\sin \omega t$ range on $(-1, 1)$ whatever the value of ω, and when one is zero the other is nonzero. So for any real numbers a and t we can choose some c_1, c_2, such that $y(t) = a$; in other words $y(t)$ spans the real numbers and so is a vector space by Theorem $\boxed{1}$ in Lay (p. 196).

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4.1.7 (20) Connections to Calculus: use properties of vector spaces to show that *continuous* real-valued functions on a closed interval are a subset of all real-valued functions on that interval

20. The set of all continuous real-valued functions on $[a, b]$ in $\mathbb{R}$ is denoted by $\mathbf{C}[a, b]$. This set is a subspace of the vector space of all real-valued functions defined on $[a, b]$.

(a) What facts about continuous functions should be proved in order to determine that $\mathbf{C}[a, b]$ is a subspace?

1. *Closed under addition:* the sum of continuous functions is continuous.
2. *Closed under scalar multiplication:* the scalar multiple of a continuous function is a continuous function.
3. *Contains the zero vector:* the function $f(x) = 0$ is a continuous function such that $g(x) + f(x) = g(x)$ for any g and any x.

(b) Show that $\{\mathbf{f} \text{ in } \mathbf{C}[a, b] : \mathbf{f}(a) = \mathbf{f}(b)\}$ is a subspace of $\mathbf{C}[a, b]$.

The class of functions that map unique inputs to the same output is the class of *non-invertible* functions.

1. *Closed under addition:* let $f(a) = f(b) = n$ and $g(a) = g(b) = m$. Now $(f + g)(a) = f(a) + g(a) = n + m$ and $(f + g)(b) = f(b) + g(b) = n + m$, and $n + m = n + m$.

2. *Closed under scalar multiplication:* $cf(a) = cn$ *and* $cf(b) = cn$ *and* $cn = cn$.

3. *Contains the zero vector:* the zero function $f(x) = 0$: $f(a) = f(b) = 0$.

4.1.8 (21–22) Apply properties of vector spaces and subspaces to sets of matrices

21. Determine if the set H of all matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is a subspace of $\boldsymbol{M}_{2\times 2}$.

1. *Closed under addition:*

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a+a & b+b \\ 0 & d+d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 0 & 2d \end{bmatrix} = 2\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

for any $\langle a, b, d\rangle$ (see [2]).

2. *Closed under scalar multiplication:* $c \cdot \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} ca & cb \\ 0 & cd \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$ which is in H for any $\langle x, y, z\rangle$.

3. *Contains the zero vector:* when $\langle a, b, d\rangle = \langle 0, 0, 0\rangle$.

4.1.9 (23–24) Support or contradict statements about the properties of vectors and vector spaces

23. Mark each statement *true* or *false* and justify the answer.

(a) If $\mathbf{f}$ is a function in the vector space V of all real-valued functions on $\mathbb{R}$ and if $\mathbf{f}(t) = 0$ for some t, then $\mathbf{f}$ is the zero vector in V.

False; $\mathbf{f}$ is a function which passes through the origin, but there may be infinitely many functions which pass through the origin. The zero vector in V is $\mathbf{g}$ such that $\mathbf{g}(t) = 0$ *for all values of* t and there is exactly one such function.

(b) A vector is an arrow in three-dimensional space.

False; we're better than this

(c) A subset H of a vector space V is a subspace of V if the zero vector is in H.

False; the vectors in H may be closed under addition and scaling *in* V but not in H; they must also be closed in H for H to be a subspace.

(d) A subspace is also a vector space.

True; any subset of a vector space inherits seven of the ten properties of vector spaces from the elements themselves, and all elements in the subset are vectors by definition of a subset of a set of vectors. The remaining three properties are implied by the definition of a subspace.

(e) Analog signals are used in the major control systems for the space shuttle, mentioned in the introduction to the chapter.

This is a weird question, but yes, the space shuttle control systems include analog signals. There may be fully-analog control loops which command analog devices with continuously-valued signals in direct response to feedback from analog sensors (also reporting continuously-valued signals). Such a system is designed directly from the physics of electromagnetics and electrochemistry.

Commonly, although perhaps not commonly at the time of the space shuttle's development, the control loop is instead modulated by a digital system; in this case the digital stage receives inputs from *ADCs (Analog-to-Digital Converters)* that receive input from analog devices (possibly after one or more filtering stages). A digital control system is designed with consideration for the dynamics of digital logic elements more than for the physics that govern analog systems.

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4.1.10 (25–30) Demonstrate understanding of properties of vector spaces by citing which are necessary to validate given claims

25. Suppose that $\vec{w}$ in V has the property that $\vec{u} + \vec{w} = \vec{w} + \vec{u} = \vec{u}$ for all $\vec{u}$ in V. In particular $\vec{0} + \vec{w} = \vec{0}$. But $\vec{0} + \vec{w} = \vec{w}$, by the *zero vector* Axiom. Hence $\vec{w} = \vec{0} + \vec{0} = \vec{0}$.

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4.1.11 (31) Show why a subspace that contains some vectors also contains their span, and why their span is the smallest subspace that contains them

31. Let $\vec{u}$ and $\vec{v}$ be vectors in a vector space V, and let H be any subspace of V that contains both $\vec{u}$ and $\vec{v}$. Explain why H also contains $\text{Span}\{\vec{u}, \vec{v}\}$. This shows that $\text{Span}\{\vec{u}, \vec{v}\}$ is the smallest subspace of V that contains both $\vec{u}$ and $\vec{v}$.

If H is a subspace then it must be true that scalar multiplication of vectors is closed in H—meaning any vectors in H scaled by any real number c produce vectors that are are also in H. Additionally the sum of any vectors in H is also in H because subspaces are closed under vector addition.

The span of $\vec{u}$ and $\vec{v}$ is the set of all linear combinations of $\vec{u}$ and $\vec{v}$, meaning any combination of the form $a\vec{u} + b\vec{v} = \vec{w}$; the set of all $\vec{w}$ in this case is $\text{Span}\,\{\vec{u}, \vec{v}\}$. We can see that the span comprises vectors in H combined by vector addition and scalar multiplication—both operators that are closed in H. So any vector $\vec{w}$ in $\text{Span}\,\{\vec{u}, \vec{v}\}$ is also in H.

4.1.12 (35–36) Use a calculator to determine if a vector is in the subspace spanned by some other vectors

35. Show that $\vec{w}$ is in the subspace of $\mathbb{R}^4$ spanned by $\vec{v}_1, \vec{v}_2, \vec{v}_3$, where

$$\vec{w} = \begin{bmatrix} 9 \\ -4 \\ -4 \\ 7 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 8 \\ -4 \\ -3 \\ 9 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -4 \\ 3 \\ -2 \\ -8 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -7 \\ 6 \\ -5 \\ -18 \end{bmatrix}$$

We're asked to find a linear combination of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ that produces the vector $\vec{w}$. In parametric form,

$$\vec{w} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3$$

If a solution exists then the vector $\vec{x} = \langle x_1, x_2, x_3 \rangle$ is the solution to the matrix equation

$$\boldsymbol{A}\vec{x} = \vec{w}$$

where $\boldsymbol{A} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$. We solve $\vec{x}$ by augmenting $\boldsymbol{A}$ with $\vec{x}$ and taking the reduced echelon form:

$$\left[\begin{array}{rrr|r} 8 & -4 & -7 & 9 \\ -4 & 3 & 6 & -4 \\ -3 & -2 & -5 & -4 \\ 9 & -8 & -18 & 7 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

In echelon form we can see that $\vec{x} = \langle 1, -2, 1, 0 \rangle$, meaning $\vec{w}$ is the combination $\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3$. The final row is $\langle 0, 0, 0, 0 \rangle$ because there is no fourth vector $\vec{v}_4$ to have a weight in $\vec{x}$; so x_4 can have any value and it does not determine the value of any component in $\vec{w}$. Consequently $\vec{w}$ is in the subspace of $\mathbb{R}^4$ spanned by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, but that subspace is isomorphic to $\mathbb{R}^3$ because the fourth degree of freedom is degenerate (there is no vector in the subspace to which the fourth coordinate can map).

4.2 Null Spaces, Column Spaces, and Linear Transformations

Exercises to practice

1–33, 35, 37–39

4.2.1 (1–2) Determine if the given vector is in the null space of the given matrix

1. Determine if $\vec{w} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ is in Nul $\boldsymbol{A}$, where $\boldsymbol{A} = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}$

We're asked to determine if the vector $\vec{w}$ is in the set of vectors mapped to the zero vector by the transformation $T(\vec{x}) = \boldsymbol{A}\vec{x}$ (the *kernel* of T). In other words, we're asked to find a set of solutions for $\vec{x}$ such that $\boldsymbol{A}\vec{x} = \vec{0}$—this is Nul $\boldsymbol{A}$—and to test if $\vec{w} \in \vec{x}$. We solve $\vec{x}$ by augmenting and taking the reduced echelon form:

$$\begin{bmatrix} \boldsymbol{A} & \vec{0} \end{bmatrix} = \left[\begin{array}{ccc|c} 3 & -5 & -3 & 0 \\ 6 & -2 & 0 & 0 \\ -8 & 4 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 1/4 & 0 \\ 0 & 1 & 3/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_3 &= s \\ x_1 &= -1/4 x_3 = -1/4 s \\ x_2 &= -3/4 x_3 = -3/4 s \end{aligned}$$

Questions to ask

So

$$\text{Nul}\,\boldsymbol{A} = \boldsymbol{A}s \begin{bmatrix} -1/4 \\ -3/4 \\ 1 \end{bmatrix}$$

where s is any real number. If $\vec{w} \in \text{Nul}\,\boldsymbol{A}$ then there is some s such that

$$\vec{w} = s \begin{bmatrix} -1/4 \\ -3/4 \\ 1 \end{bmatrix}$$

We solve for s algebraically on a per-row basis,

$$\vec{w} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -1/4 s \\ -3/4 s \\ s \end{bmatrix} = \begin{bmatrix} -1/4(-4) \\ -3/4(-4) \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$$

and we see that $\vec{w}$ is in Nul $\boldsymbol{A}$.

4.2.2 (3–6) Describe the null space of the given matrix by finding vectors that span the null space

In exercises 3–6, find an explicit description of Nul $\boldsymbol{A}$ by listing vectors that span the null space.

3. $\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$

Nul $\boldsymbol{A}$ = Span $\{\vec{x}\}$ where $\boldsymbol{A}\vec{x} = \vec{0}$

$$\begin{bmatrix}\boldsymbol{A} & \vec{0}\end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 3 & 5 & 0 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{array}\right] \overset{\text{RREF}}{\longrightarrow} \left[\begin{array}{cccc|c} 1 & 0 & -7 & 6 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{array}\right]$$

$$\begin{aligned} x_1 - 7x_3 + 6x_4 &= 0 \\ x_2 + 4x_3 - 2x_4 &= 0 \\ x_3 &= s \\ x_4 &= t \end{aligned}$$

We see that the third and fourth coordinates are independent: $\boldsymbol{A}$ maps a plane through the origin to the zero vector in $\mathbb{R}^4$ (the origin is included when $s = t = 0$). The plane is in $\mathbb{R}^4$ because $\boldsymbol{A}$ is a mapping from a space with two coordinates to a space with four coordinates.

$$\begin{aligned} x_1 &= 7s - 6t \\ x_2 &= -4s + 2t \end{aligned}$$

$$\text{Span}\{\vec{x}\} = \text{Nul}\,\boldsymbol{A} = \begin{bmatrix} 7s - 6t \\ -4s + 2t \\ s \\ t \end{bmatrix} \text{ for all real numbers } s, t$$

We list two vectors in Nul $\boldsymbol{A}$: when $\langle s, t\rangle = \langle 0, 1\rangle$ and $\langle 1, 0\rangle$, respectively:

$$\begin{bmatrix} 7(0) - 6(1) \\ -4(0) + 2(1) \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 7(1) - 6(0) \\ -4(1) + 2(0) \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

4.2.3 (7–14) Use properties of vector spaces to prove that the given set is a vector space, or give a specific example proving it is not

Use an appropriate theorem to show that the set W is a vector space, or find a specific example to the contrary.

7. $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a+b+c=2 \right\}$

W is not a vector space because it does not contain the zero vector. There is no $\langle a,b,c\rangle$ such that

$$a+b+c=2 \quad (1)$$

that also satisfies

$$\langle a,b,c\rangle + \langle x,y,z\rangle = \langle x,y,z\rangle \quad (2)$$

The only vectors that satisfy (1) have at least one nonzero entry:

$$\langle 2,0,0\rangle, \langle 0,2,0\rangle, \langle 0,0,2\rangle$$

Any such vector cannot satisfy (2), so there is no zero vector in W.

☐☐☐ 4.2.4 (15–16) Work backward to construct a matrix that has the given column space

Find $\boldsymbol{A}$ such that the given set is Col $\boldsymbol{A}$

15. $\left\{ \begin{bmatrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-s-t \end{bmatrix} : r,s,t \in \mathbb{R} \right\}$

By definition, the column space of $\boldsymbol{A}$ is the set of all linear combinations of the columns of $\boldsymbol{A}$ (Lay p. 203). This question asks us to find some vectors such that all their linear combinations are described by the given set.

Vectors in the set are linear combinations of three vectors with weights r, s, and t, described by the parametric form

$$r \begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ 0 \\ -1 \end{bmatrix}$$

We let the weights r,s,t be the respective entries $\langle x_1, x_2, x_3\rangle$ in a vector $\vec{x}$ and rewrite the parametric set as the matrix equation

$$\boldsymbol{A}\vec{x} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{b}$$

Now any $\vec{b}$ in the above equation is a linear combination of the columns of $\boldsymbol{A}$ with weights given as $\vec{x}$. The set of all vectors $\vec{b}$ solving for all weights $\vec{x}$ is Col $\boldsymbol{A}$.

4.2.5 (17–20) Identify the dimension of the null and column spaces of the given matrix

For the matrix $\boldsymbol{A}$, (a) find k such that $\operatorname{Nul}\boldsymbol{A}$ is a subspace of $\mathbb{R}^k$, and (b) find k such that $\operatorname{Col}\boldsymbol{A}$ is a subspace of $\mathbb{R}^k$.

17. $\boldsymbol{A} = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$

The columns of $\boldsymbol{A}$ are linearly dependent, so $\dim\operatorname{Col}\boldsymbol{A} = 1$ and $\operatorname{Col}\boldsymbol{A}$ is a one-dimensional subspace of $\mathbb{R}^4$ (there is one degree of freedom to specify all vectors in the span of the columns because they span the same vectors). *Note that* $\operatorname{Col}\boldsymbol{A}$ *is not a subspace of* $\mathbb{R}^1$ *because its vectors have four entries.*

To find $\operatorname{Nul}\boldsymbol{A}$ we augment and row reduce:

$$\left[\begin{array}{rr|r} 2 & -6 & 0 \\ -1 & 3 & 0 \\ -4 & 12 & 0 \\ 3 & -9 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rr|r} 1 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

There are three free variables in $\operatorname{Nul}\boldsymbol{A}$ and one dependent variable. $\operatorname{Nul}\boldsymbol{A}$ is therefore a one-dimensional subspace of $\mathbb{R}^2$.

4.2.6 (21–22) Find specific vectors in the null and column spaces of the given matrix

21. Considering the matrix $\boldsymbol{A}$ from Exercise 17 on this page, find a nonzero vector in $\operatorname{Nul}\boldsymbol{A}$ and a nonzero vector in $\operatorname{Col}\boldsymbol{A}$.

$\operatorname{Col}\boldsymbol{A}$ is the set of vectors spanned by the columns of $\boldsymbol{A}$; in parametric form,

$$\operatorname{Col}\boldsymbol{A} = \left\{ t \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix} + s \begin{bmatrix} -6 \\ 3 \\ 12 \\ -9 \end{bmatrix} : t, s \in \mathbb{R} \right\}$$

However, consider when $\langle t, s\rangle = \langle 3, 1\rangle$:

$$3 \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \\ 12 \\ -9 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -12 \\ 9 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \\ 12 \\ -9 \end{bmatrix} = \vec{0}$$

These columns are not linearly independent: they span the same vectors. The column space is therefore the set of vectors when $s = 0$ (or when $t = 0$):

$$\text{Col}\,\boldsymbol{A} = \left\{ t \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix} : t \in \mathbb{R} \right\}$$

and one vector from the column space is given when $t = 1$:

$$\begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix} \in \text{Col}\,\boldsymbol{A}$$

We find the equation for vectors in the null space of A by augmenting $\begin{bmatrix} \boldsymbol{A} & \vec{0} \end{bmatrix}$ and taking the reduced echelon form to identify solutions for $\vec{x}$ in $\boldsymbol{A}\vec{x} = \vec{0}$. Borrowing from Exercise 17, that is

$$\left[\begin{array}{rr|r} 2 & -6 & 0 \\ -1 & 3 & 0 \\ -4 & 12 & 0 \\ 3 & -9 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{rr|r} 1 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 = 3t$$
$$x_2 = t$$

Vectors in Nul $\boldsymbol{A}$ are then solutions for $\vec{x} = \langle x_1, x_2 \rangle$:

$$\text{Nul}\,\boldsymbol{A} = \left\{ t \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$$

and one vector from the null space is again given when $t = 1$:

$$\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in \text{Nul}\,\boldsymbol{A}$$

☐☐☐ 4.2.7 (23–24) Determine if the given vector is in the null or column spaces of the given matrix

23. Let $\boldsymbol{A} = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Determine if $\vec{w}$ is in Col $\boldsymbol{A}$ and if $\vec{w}$ is in Nul $\boldsymbol{A}$.

At a glance the columns of $\boldsymbol{A}$ are linearly dependent; we confirm this by observing the free variable in the reduced row echelon form:

$$\boldsymbol{A} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

We describe Col $\boldsymbol{A}$ by writing $\boldsymbol{A}$ in parametric form:

$$\text{Col}\,\boldsymbol{A} = \text{Span}\left\{ t \begin{bmatrix} -6 \\ -3 \end{bmatrix} : t \in \mathbb{R} \right\}$$

and find a solution for t such that $t \begin{bmatrix} -6 \\ -3 \end{bmatrix} = \vec{w}$:

$$\left[\begin{array}{c|c} -6 & 2 \\ -3 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{c|c} -6 & 2 \\ -3 & 1 \end{array}\right] = \left[\begin{array}{c|c} 1 & -1/3 \\ 0 & 0 \end{array}\right]$$

so $\vec{w}$ is in Col $\boldsymbol{A}$ when $t = -1/3$.

4.2.8 (25–26) Support or contradict statements about the null and column spaces of an $m \times n$ matrix

25. (a) The null space of $\boldsymbol{A}$ is the solution set of the equation $\boldsymbol{A}\vec{x} = \vec{0}$.

True

(b) The null space of an $m \times n$ matrix is in $\mathbb{R}^m$.

False; the null space is the set of solutions $\vec{x}$ in $\boldsymbol{A}\vec{x} = \vec{0}$; that multiplication is only defined when $\vec{x}$ has as many rows as $\boldsymbol{A}$ has columns. The vectors in the null space are therefore vectors in $\mathbb{R}^n$.

(c) The column space of $\boldsymbol{A}$ is the range of the mapping $\vec{x} \mapsto \boldsymbol{A}\vec{x}$.

True; the column space of $\boldsymbol{A}$ is all the linear combinations of the columns of $\boldsymbol{A}$. One way to generate those combinations is to multiply the $m \times n$ matrix $\boldsymbol{A}$ by a column vector $\vec{x}$ of n rows; each column is weighted with a value in $\vec{x}$ and the columns are summed to give the row of the resultant vector—a linear combination of the columns of $\boldsymbol{A}$. So each $\vec{x}$ gives a unique combination of columns in $\boldsymbol{A}$, and the set of all those combinations is the column space of $\boldsymbol{A}$.

(d) If the equation $\boldsymbol{A}\vec{x} = \vec{b}$ is consistent, then Col $\boldsymbol{A}$ is $\mathbb{R}^m$.

False; only if the equation is consistent *for all* $\vec{b}$*;* then there is a solution $\vec{b}$ for every $\vec{x}$ and transformation by $\boldsymbol{A}$ is a surjection from $\mathbb{R}^m$ onto $\mathbb{R}^m$.

(e) The kernel of a linear transformation is a vector space.

True; when the linear transformation is a matrix transformation, the kernel is the null space of its standard matrix. It contains all the vectors whose image under transformation is the zero vector.

(f) Col $\boldsymbol{A}$ is the set of all vectors that can be written as $\boldsymbol{A}\vec{x}$ for some $\vec{x}$.

True; see (c): the column space is the range (output) of the mapping $\vec{x} \mapsto \boldsymbol{A}\vec{x}$.

4.2.9 (27–28) Use properties of null and column spaces to prove relationships between systems of linear equations

27. It can be shown that a solution of the system below is $x_1 = 3$, $x_2 = 2$, and $x_3 = -1$. Use this fact and the theory from this section to explain why another solution is $x_1 = 30$, $x_2 = 20$, and $x_3 = -10$ without making any calculations.

$$\begin{aligned} x_1 - 3x_2 - 3x_3 &= 0 \\ -2x_1 + 4x_2 + 2x_3 &= 0 \\ -x_1 + 5x_2 + 7x_3 &= 0 \end{aligned}$$

The solutions for $\vec{x}$ in this system are vectors in the null space of the matrix of coefficients on x_1, x_2, x_3 (call that matrix $\boldsymbol{A}$). That is, the vectors $\{\vec{x}\}$ are mapped to the zero vector by the linear transformation described by this system, written as $\vec{x} \mapsto \boldsymbol{A}\vec{x} = 0$. The key to answering this question is to observe that the null space of $\boldsymbol{A}$ is a vector space (specifically a subspace of $\mathbb{R}^3$); the implication being that the solutions for $\vec{x}$ have the properties of vectors. Vector spaces (and subspaces) are closed under scalar multiplication, so scalar multiples of $\vec{x}$ are also in Nul $\boldsymbol{A}$ and are solutions to the system.

We're given that one of the solutions for $\vec{x}$ is $\langle 3, 2, -1\rangle$, so that vector is—and all its scalar multiples are—in Nul $\boldsymbol{A}$. Conversely all the vectors in Nul $\boldsymbol{A}$ are also solutions for $\vec{x}$. Nul $\boldsymbol{A}$ is a vector space so it is closed under scalar multiplication. A scalar multiple of $\langle 3, 2, -1\rangle$ is $10 \cdot \langle 3, 2, -1\rangle = \langle 30, 20, -10\rangle$, which is also in Nul $\boldsymbol{A}$ and so is a solution for $\vec{x}$ in the system above.

4.2.10 (29) Prove that a column space is a subspace of $\mathbb{R}^n$

Theorem 3 (Lay p. 203) states that "the column space of an $m \times n$ matrix $\boldsymbol{A}$ is a subspace of $\mathbb{R}^m$.

29. Prove Theorem 3 as follows: Given an $m \times n$ matrix $\boldsymbol{A}$, an element in Col $\boldsymbol{A}$ has the form $\boldsymbol{A}\vec{x}$ for some $\vec{x}$ in $\mathbb{R}^3$. Let $\boldsymbol{A}\vec{x}$ and $\boldsymbol{A}\vec{w}$ represent any two vectors in Col $\boldsymbol{A}$.

(a) Explain why the zero vector is in Col $\boldsymbol{A}$.

Col $\boldsymbol{A}$ contains the zero vector at least when $\vec{x} = \vec{0}$ (the trivial solution for $\vec{x}$ in $\boldsymbol{A}\vec{x} = \vec{0}$).

(b) Show that the vector $\boldsymbol{A}\vec{x} + \boldsymbol{A}\vec{w}$ is in Col $\boldsymbol{A}$.

Let the columns of $\boldsymbol{A}$ be $\{\vec{v}_1, \ldots, \vec{v}_n\}$. Then $\boldsymbol{A}\vec{x} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$ and $\boldsymbol{A}\vec{w} = w_1\vec{v}_1 + \cdots + w_n\vec{v}_n$. $\boldsymbol{A}\vec{x} + \boldsymbol{A}\vec{w}$ expands to $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n + w_1\vec{v}_1 + \cdots + w_n\vec{v}_n$; factoring, we have $(x_1 + w_1)\,\vec{v}_1 + \cdots + (x_n + w_n)\,\vec{v}_n$. Let $c_i = x_i + w_i$ and substitute:

$$\begin{aligned}&(x_1 + w_1)\,\vec{v}_1 + \cdots + (x_n + w_n)\,\vec{v}_n\\ &= c_1\vec{v}_1 + \cdots + c_n\vec{v}_n\\ &= \boldsymbol{A}\vec{c}\end{aligned}$$

$\vec{c} \in \mathbb{R}^3$ so $\boldsymbol{A}\vec{c} = \boldsymbol{A}\vec{x} + \boldsymbol{A}\vec{w}$ is in Col $\boldsymbol{A}$.

(c) Given a scalar c, show that $c(\boldsymbol{A}\vec{x})$ is in Col $\boldsymbol{A}$.

Again let the columns of $\boldsymbol{A}$ be $\{\vec{v}_1, \ldots, \vec{v}_n\}$ so that $\boldsymbol{A}\vec{x} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$; then $c(\boldsymbol{A}\vec{x}) = cx_1\vec{v}_1 + \cdots + cx_n\vec{v}_n$. Let $y_i = cx_i$, so $c(\boldsymbol{A}\vec{x}) = y_1\vec{v}_1 + \cdots + y_n\vec{v}_n$. Now $y_i \in \mathbb{R}^3$, so $c(\boldsymbol{A}\vec{x}) = \boldsymbol{A}\vec{y} \in$ Col $\boldsymbol{A}$.

☐☐☐ 4.2.11 (30) Prove that a linear transformation $T : V \to W$ maps a subspace of W

30. Let $T : V \to W$ be a linear transformation from a vector space V into a vector space W. Prove that the range of T is a subspace of W.

A linear transformation from V to W is a mapping of vectors $\vec{v}$ in V to vectors $T(\vec{v})$ in W (see definition, Lay p. 206). From the definition of a linear transformation (Lay p. 66) we have two properties that relate the transformation to a subspace:

- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in V
- $T(c\vec{u}) = cT(\vec{u})$ for all scalars c and all $\vec{u}$ in V

From the second it follows that the zero vector in V is mapped by T: let $c = 0$, then $cT(\vec{u}) = T(0 \cdot \vec{u}) = T(\vec{0})$ for every $\vec{u}$ in V (see Equation 1 in Lay p. 193). Together with the zero vector mapping, the above two rules are sufficient to show that T maps a subspace: both addition and scalar multiplication of images under transformation produce vectors which are the images of other transformations.

4.2.12 (31–32) Prove that $T : \mathbb{P}_3 \to \mathbb{R}^2$ is a linear transformation and describe its kernel and range

31. Define $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\vec{p}) = \begin{bmatrix} \vec{p}(0) \\ \vec{p}(1) \end{bmatrix}$. For instance, if $\vec{p}(t) = 3 + 5t + 7t^2$, then $T(\vec{p}) = \begin{bmatrix} 3 \\ 15 \end{bmatrix}$.

(a) Show that T is a linear transformation.

Let $\vec{p}, \vec{q}$ be some polynomials in $\mathbb{P}_2$. T is a linear transformation if $T(\vec{p} + \vec{q}) = T(\vec{p}) + T(\vec{q})$ and $cT(\vec{p}) = T(c\vec{p})$. We handle the addition case first:

$$T(\vec{p}) = \begin{bmatrix} \vec{p}(0) \\ \vec{p}(1) \end{bmatrix}, T(\vec{q}) = \begin{bmatrix} \vec{q}(0) \\ \vec{q}(1) \end{bmatrix}$$

$$\begin{aligned} T(\vec{p} + \vec{q}) = \begin{bmatrix} (\vec{p} + \vec{q})(0) \\ (\vec{p} + \vec{q})(1) \end{bmatrix} &= \begin{bmatrix} (p_1 + q_1) + (p_2 + q_2) \cdot 0 + (p_3 + q_3) \cdot 0^2 \\ (p_1 + q_1) + (p_2 + q_2) \cdot 1 + (p_3 + q_3) \cdot 1^2 \end{bmatrix} \\ &= \begin{bmatrix} p_1 + q_1 \\ p_1 + p_2 + p_3 + q_1 + q_2 + q_3 \end{bmatrix} \\ T(\vec{p}) + T(\vec{q}) = \begin{bmatrix} \vec{p}(0) \\ \vec{p}(1) \end{bmatrix} + \begin{bmatrix} \vec{q}(0) \\ \vec{q}(1) \end{bmatrix} &= \begin{bmatrix} \vec{p}(0) + \vec{q}(0) \\ \vec{p}(1) + \vec{q}(1) \end{bmatrix} \\ &= \begin{bmatrix} \left(p_1 + p_2 \cdot 0 + p_3 \cdot 0^2\right) + \left(q_1 + q_2 \cdot 0 + q_3 \cdot 0^2\right) \\ \left(p_1 + p_2 \cdot 1 + p_3 \cdot 1^2\right) + \left(q_1 + q_2 \cdot 1 + q_3 \cdot 1^2\right) \end{bmatrix} \\ &= \begin{bmatrix} p_1 + q_1 \\ p_1 + p_2 + p_3 + q_1 + q_2 + q_3 \end{bmatrix} \end{aligned}$$

The result is identical whatever order the operations are formed in, so T is linear in addition. For scalar multiplication we see the same result:

$$\begin{aligned} cT(\vec{p}) = c \cdot \begin{bmatrix} \vec{p}(0) \\ \vec{p}(1) \end{bmatrix} = \begin{bmatrix} c \cdot \vec{p}(0) \\ c \cdot \vec{p}(1) \end{bmatrix} &= \begin{bmatrix} c\left(p_1 + p_2 \cdot 0 + p_3 \cdot 0^2\right) \\ c\left(p_1 + p_2 \cdot 1 + p_3 \cdot 1^2\right) \end{bmatrix} \\ &= \begin{bmatrix} c \cdot p_1 \\ c \cdot p_1 + c \cdot p_2 + c \cdot p_3 \end{bmatrix} \\ T(c\vec{p}) = \begin{bmatrix} (c \cdot \vec{p})(0) \\ (c \cdot \vec{p})(1) \end{bmatrix} &= \begin{bmatrix} c \cdot p_1 + (c \cdot p_2) \cdot 0 + (c \cdot p_3) \cdot 0^2 \\ c \cdot p_1 + (c \cdot p_2) \cdot 1 + (c \cdot p_3) \cdot 1^2 \end{bmatrix} \\ &= \begin{bmatrix} c \cdot p_1 \\ c \cdot p_1 + c \cdot p_2 + c \cdot p_3 \end{bmatrix} \end{aligned}$$

so T is a linear transformation.

(b) Find a polynomial $\vec{p}$ in $\mathbb{P}_2$ that spans the kernel of T, and describe the range of T.

The kernel of T is all the vectors that T maps to the zero vector. That is, any $\vec{p} \in \mathbb{P}_2$ such that $T(\vec{p}) = \begin{bmatrix} \vec{p}(0) \\ \vec{p}(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We write out $\vec{p}(0) = 0$ and $\vec{p}(1) = 0$ with unknown coefficients p_i,

$$\vec{p}(0) = p_1 + p_2 \cdot 0 + p_3 \cdot 0^2 = p_1 = 0$$
$$\vec{p}(1) = p_1 + p_2 \cdot 1 + p_3 \cdot 1^2 = p_2 + p_3 = 0$$

Substituting s for p_3, we have $\langle p_1, p_2, p_3 \rangle = \langle 0, -s, s \rangle$. The kernel of T then is all polynomials of the form $\vec{p}(t) = -st + st^2 = st(t-1)$ where $s \in \mathbb{R}$.

The range of T is $\mathbb{R}^2$.

4.2.13 (33) Apply properties of linear transformations to a transformation between vector spaces of matrices

33. Let $M_{2\times 2}$ be the vector space of all 2×2 matrices, and define $T : M_{2\times 2} \to M_{2\times 2}$ by $T(\boldsymbol{A}) = \boldsymbol{A} + \boldsymbol{A}^T$, where $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) Show that T is a linear transformation.

Linear addition,

$$\begin{aligned} T(\boldsymbol{A}+\boldsymbol{B}) &= \boldsymbol{A}+\boldsymbol{B}+(\boldsymbol{A}+\boldsymbol{B})^T \\ T(\boldsymbol{A})+T(\boldsymbol{B}) &= \boldsymbol{A}+\boldsymbol{A}^T+\boldsymbol{B}+\boldsymbol{B}^T \\ &= \boldsymbol{A}+\boldsymbol{B}+\boldsymbol{A}^T+\boldsymbol{B}^T && \text{Theorem } \boxed{1}\text{, Lay p. 95} \\ &= \boldsymbol{A}+\boldsymbol{B}+(\boldsymbol{A}+\boldsymbol{B})^T && \text{Theorem } \boxed{3}\text{, Lay p. 101} \\ T(\boldsymbol{A})+T(\boldsymbol{B}) &= T(\boldsymbol{A}+\boldsymbol{B}) \end{aligned}$$

and linear scalar multiplication:

$$\begin{aligned} &cT(\boldsymbol{A}) = c\left(\boldsymbol{A}+\boldsymbol{A}^T\right) = c\boldsymbol{A}+c\boldsymbol{A}^T \\ &T(c\boldsymbol{A}) = c\boldsymbol{A}+(c\boldsymbol{A})^T = c\boldsymbol{A}+c\boldsymbol{A}^T && \text{Theorem } \boxed{3}\text{, Lay p. 101} \\ &T(c\boldsymbol{A}) = cT(\boldsymbol{A}) \end{aligned}$$

(b) Let $\boldsymbol{B}$ be any element of $M_{2\times 2}$ such that $\boldsymbol{B}^T = \boldsymbol{B}$. Find an $\boldsymbol{A}$ in $M_{2\times 2}$ such that $T(\boldsymbol{A}) = \boldsymbol{B}$.

$$T(\boldsymbol{A}) = \boldsymbol{A}+\boldsymbol{A}^T = \boldsymbol{B} = \boldsymbol{B}^T$$
$$\boldsymbol{B}+\boldsymbol{B}^T = 2\boldsymbol{B}$$

$$T\left(\frac{1}{2}\boldsymbol{B}\right) = \frac{1}{2}\boldsymbol{B} + \frac{1}{2}\boldsymbol{B}^T = \boldsymbol{B}$$

$$\boldsymbol{A} = \frac{1}{2}\boldsymbol{B}$$

(c) Show that the range of T is the set of $\boldsymbol{B}$ in $M_{2\times 2}$ with the property that $\boldsymbol{B}^T = \boldsymbol{B}$.

$$\begin{aligned}\boldsymbol{B} = T(\boldsymbol{A}) &= \boldsymbol{A} + \boldsymbol{A}^T \\ \boldsymbol{B}^T = T(\boldsymbol{A})^T &= \left(\boldsymbol{A} + \boldsymbol{A}^T\right)^T \\ &= \boldsymbol{A}^T + \boldsymbol{A}^{TT} = \boldsymbol{A}^T + \boldsymbol{A} \\ &= \boldsymbol{B}\end{aligned}$$

It appears that $\boldsymbol{B} = \boldsymbol{B}^T$ where $\boldsymbol{B} = T(\boldsymbol{A})$ for any $\boldsymbol{A}$ in $M_{2\times 2}$.

(d) Describe the kernel of T.

$$\begin{aligned}T(\boldsymbol{A}) = \boldsymbol{A} + \boldsymbol{A}^T &= \boldsymbol{0} \\ \boldsymbol{A}^T &= -\boldsymbol{A}\end{aligned}$$

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \boldsymbol{A}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$$

This equality holds when

$$\begin{aligned}a &= -a = 0 \\ d &= -d = 0 \\ c &= -b \\ b &= -c = b\end{aligned}$$

The kernel of T is the set of 2×2 matrices $\left\{\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}\right\}$.

4.2.14 (35) Observe the relationship between a linear transformation and subspaces of its codomain

35. Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Given a subspace U of V, let $T(U)$ denote the set of all images of the form $T(\vec{x})$, where $\vec{x}$ is in U. Show that $T(U)$ is a subspace of W.

T is a linear transformation, so it's given that $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ and $cT(\vec{x}) = T(c\vec{x})$. By these properties of linear transformations we have the following conclusions:

- For any vectors $\vec{x}$ and $\vec{y}$ both in U, $T(\vec{x})$ and $T(\vec{y})$ are both in $T(U)$ so $T(\vec{x}+\vec{y})$ is also in $T(U)$ and $T(U)$ is closed under addition.
- For any $\vec{x} \in U$ where $T(\vec{x}) \in T(U)$, $c\vec{x}$ must also be in U (by properties of vectors) so $T(c\vec{x}) = cT(\vec{x}) \in T(U)$, so $T(U)$ is closed under scalar multiplication.
- If $T(c\vec{x}) \in T(U)$ then $T(0 \cdot \vec{x}) \in T(U)$ for any $\vec{x}$, so the zero vector is in $T(U)$.

Those three conclusions are sufficient to determine that $T(U)$ is a subspace. Because T maps vectors from V to W, all $T(U)$ are in W so $T(U)$ is a subspace of W.

4.2.15 (37–38) Use a calculator to determine if a vector is in the null or column spaces of the given matrix

37. Determine whether $\vec{w}$ is in the column space of $\boldsymbol{A}$, the null space of $\boldsymbol{A}$, or both, where

$$\vec{w} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix}, \quad A = \begin{bmatrix} 7 & 6 & -4 & 1 \\ -5 & -1 & 0 & -2 \\ 9 & -11 & 7 & -3 \\ 19 & -9 & 7 & 1 \end{bmatrix}$$

$\vec{w}$ is in Col $\boldsymbol{A}$ if there exists a solution for $\vec{x}$ in $\boldsymbol{A}\vec{x} = \vec{w}$. Augment and row reduce:

$$\begin{bmatrix} \boldsymbol{A} & \vec{w} \end{bmatrix} = \left[\begin{array}{cccc|c} 7 & 6 & -4 & 1 & 1 \\ -5 & -1 & 0 & -2 & 1 \\ 9 & -11 & 7 & -3 & -1 \\ 19 & -9 & 7 & 1 & -3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/95 & 1/95 \\ 0 & 1 & 0 & 39/95 & -20/19 \\ 0 & 0 & 1 & 267/95 & -172/95 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Not only is $\vec{w}$ in Col $\boldsymbol{A}$, but only the first three columns are needed to make $\vec{w}$. To find if $\vec{w} \in$ Nul $\boldsymbol{A}$ we need a description of Nul $\boldsymbol{A}$, so we again augment and row reduce:

$$\begin{bmatrix} \boldsymbol{A} & \vec{0} \end{bmatrix} = \left[\begin{array}{cccc|c} 7 & 6 & -4 & 1 & 0 \\ -5 & -1 & 0 & -2 & 0 \\ 9 & -11 & 7 & -3 & 0 \\ 19 & -9 & 7 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/95 & 0 \\ 0 & 1 & 0 & 39/19 & 0 \\ 0 & 0 & 1 & 267/95 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

One variable, x_4, is free so we describe Nul $\boldsymbol{A}$ in terms of it:

$$\text{Nul}\,\boldsymbol{A} = \left\{ \begin{bmatrix} (1/95)\,s \\ (-39/19)\,s \\ (-267/95)\,s \\ s \end{bmatrix} : s \in \mathbb{R} \right\}$$

Now we find some s such that $s \cdot \vec{w} \in \operatorname{Nul} \boldsymbol{A}$ according to the description above

$$\begin{bmatrix} 1/95 & 1 \\ -39/19 & 1 \\ -267/95 & -1 \\ 1 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This appears to say that $\begin{bmatrix} (1/95) \\ (-39/19) \\ (-267/95) \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix}$ which is false, so $\vec{w}$ is not in $\operatorname{Nul} \boldsymbol{A}$.

4.2.16 (39) Use a calculator to find a set of vectors that span the null space of the given matrix; use definitions of column spaces to prove relationships between the matrix and some given vectors

39. Let $\vec{a}_1, \ldots, \vec{a}_5$ denote the columns of the matrix $\boldsymbol{A}$, where

$$\boldsymbol{A} = \begin{bmatrix} 5 & 1 & 2 & 2 & 0 \\ 3 & 3 & 2 & -1 & -12 \\ 8 & 4 & 4 & -5 & 12 \\ 2 & 1 & 1 & 0 & -2 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_4 \end{bmatrix}$$

(a) Explain why $\vec{a}_3$ and $\vec{a}_5$ are in the column space of $\boldsymbol{B}$.

This would be the case if $\vec{a}_3$ and $\vec{a}_5$ were not linearly independent columns of $\boldsymbol{A}$. We check that by row reducing:

$$\boldsymbol{A} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1/3 & 0 & 10/3 \\ 0 & 1 & 1/3 & 0 & -26/3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Indeed we see there is no pivot position for the third or fifth columns of $\boldsymbol{A}$, so $\vec{a}_3$ and $\vec{a}_5$ are not linearly independent in $\boldsymbol{A}$ so are in the span of $\{\vec{a}_1, \vec{a}_3, \vec{a}_4\}$.

(b) Find a set of vectors that spans $\operatorname{Nul} \boldsymbol{A}$.

Augment and row reduce to find a description of $\operatorname{Nul} \boldsymbol{A}$ (note: this is the reduced form found in part [a]):

$$\begin{bmatrix} \boldsymbol{A} & \vec{0} \end{bmatrix} \xrightarrow{\text{RREF}} \left[\begin{array}{ccccc|c} 1 & 0 & 1/3 & 0 & 10/3 & 0 \\ 0 & 1 & 1/3 & 0 & -26/3 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We describe Nul $\boldsymbol{A}$ in parametric form, substituting $t = x_3$ and $s = x_5$:

$$\text{Nul}\,\boldsymbol{A} = \left\{ t \begin{bmatrix} -1/3 \\ -1/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -10/3 \\ 26/3 \\ 0 \\ 4 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

(c) Let $T : \mathbb{R}^5 \to \mathbb{R}^4$ be defined by $T(\vec{x}) = \boldsymbol{A}\vec{x}$. Explain why T is neither *one-to-one* nor *onto*.

T cannot be *one-to-one* or *onto* because there are only three independent vectors in the columns of its standard matrix $\boldsymbol{A}$; T therefore maps to a three-dimensional subspace of $\mathbb{R}^4$. There are neither enough basis vectors in the range of T to uniquely map each vector in $\mathbb{R}^5$ (so T is not *one-to-one*); nor are there enough to represent all the vectors in $\mathbb{R}^4$ as linear combinations of them (so T is not *onto*).

4.3 Linearly Independent Sets; Bases

1–25, 33, 34, 37

Exercises to practice

☐☐☐

4.3.1 (1–8) Identify bases of $\mathbb{R}^3$; show that linear independence and spanning are necessary properties of a basis

Determine which sets of vectors are bases for $\mathbb{R}^3$. Of the sets that are *not* bases, determine which ones are linearly independent and which ones span $\mathbb{R}^3$ and justify these determinations.

1. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

If we create a matrix $\boldsymbol{A}$ by concatenating the given vectors as columns we see that the rows are already in echelon form and there is a pivot in each column. Because there are three rows and also three pivots, by properties of invertible matrices we can infer that

- the given vectors are linearly independent
- the linear transformation $T(\vec{x}) = \boldsymbol{A}\vec{x}$ is a surjection ($T : \mathbb{R}^3 \twoheadrightarrow \mathbb{R}^3$)
- the given vectors span $\mathbb{R}^3$

Any of these properties is sufficient to prove that the vectors are a basis for $\mathbb{R}^3$. By the *spanning set theorem* (5 in Lay p. 212) we can say that a basis for $\mathbb{R}^3$ is formed by the minimal set of the given vectors that spans $\mathbb{R}^3$; here that minimal set is all three of those vectors.

Questions to ask

☐☐☐

4.3.2 (9–10) Find as basis for the null space of the given matrix

Refer to Example 3 *in Section 4.3 (Lay pp. 202–203):*

Find a spanning set for the null space of the matrix $\boldsymbol{A}$. The first step is to find the general solution of $\boldsymbol{A}\vec{x} = \vec{0}$ in terms of free variables. Take the reduced echelon form of the augmented matrix $\begin{bmatrix}\boldsymbol{A} & \vec{0}\end{bmatrix}$ and write the basic variables in terms of the free variables. Write the resultant system in parametric form where the weights are the free variables. Every solution to this parametric equation is an element of Nul $\boldsymbol{A}$.

9. $\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix}$

Let $\boldsymbol{A}$ be the given matrix. A basis for the null space of $\boldsymbol{A}$ is a set of vectors that spans Nul $\boldsymbol{A}$. We first find a parametric equation for the elements of Nul $\boldsymbol{A}$

$$\begin{bmatrix}\boldsymbol{A} & \vec{0}\end{bmatrix} = \left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 3 & -2 & 1 & -2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_3 - 2x_4 \\ 5x_3 - 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

All the vectors in Nul $\boldsymbol{A}$ then have some coordinates $\langle x_3, x_4 \rangle$ in the equation above. The vectors in that equation then are a basis $\mathcal{B}$ for Nul $\boldsymbol{A}$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

4.3.3 (11) Find a basis on a specific plane in $\mathbb{R}^3$

11. Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x + 2y + z = 0$.

The set of vectors on a plane in $\mathbb{R}^3$ is described by

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + 2y + z = 0 \text{ and } x, y, z \in \mathbb{R} \right\}$$

. V is a set of vectors in $\mathbb{R}^3$, but it is a subspace of $\mathbb{R}^3$ and is isomorphic to $\mathbb{R}^2$; that is, there are not three independent basis vectors for V—there are two independent vectors weighted by $\langle y, z \rangle$ and one dependent vector weighted by x. The third vector does not span any vectors not spanned by the first two and so is not part of a basis (a minimal spanning set); instead it is solved in terms of the other vectors, which we see below.

We find that V is isomorphic to $\mathbb{R}^2$ when isolating a variable in the equation; whatever variable we choose we may solve for it in terms of the other two, but we can't solve more than one variable. As above we let x be the dependent variable:

$$x = -2y - z$$

Now we may write the coordinate of any vector in V as described above, but we substitute x with the variables on which it depends:

$$V = \left\{ \begin{bmatrix} -2y - z \\ y \\ z \end{bmatrix} \right\}$$

Writing this in parametric form we have weights y, z on vectors $\vec{v}_1, \vec{v}_2$ such that V is all the linear combinations $y\vec{v}_1 + z\vec{v}_2$. We let $x_1 = y$ and $x_2 = z$ and we have

$$V = x_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

It follows that these vectors span V with coordinates $\vec{x} = \langle x_1, x_2 \rangle$. As a matrix equation,

$$\boldsymbol{A}\vec{x} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{v} \in V$$

Those vectors which combined in the parametric form and which form the columns of our matrix $\boldsymbol{A}$ are a basis that span V.

4.3.4 (12) Find a basis on a specific line in $\mathbb{R}^2$

12. Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y = 5x$.

The set of vectors $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = 5x \text{ and } x, y \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^2$ that is isomorphic to $\mathbb{R}^1$. As in Exercise 11 on the previous page, we write the parametric form of this set, replacing the dependent variable with its equivalent function of the independent variable

$$V = \left\{ \begin{bmatrix} x \\ 5x \end{bmatrix} \right\}$$

$$V = \left\{ x \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\}$$

An obvious basis vector for V is $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$. This exercise spurs some interesting imaginations about the relationship between this basis vector and the derivative of the function $y = 5x$—consider that all basis vectors for V have the same angle (or its inverse) and how that angle is related to $y\prime = 5$.

4.3.5 (13–14) Use two row equivalent matrices to find bases for the null and column spaces of one of them

Assume that $\boldsymbol{A}$ is row equivalent to $\boldsymbol{B}$. Find bases for Nul $\boldsymbol{A}$ and Col $\boldsymbol{A}$.

13. $\boldsymbol{A} = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Row equivalence gives us that any column dependence in $\boldsymbol{B}$ must also exist in $\boldsymbol{A}$. We see that $\boldsymbol{B}$ has two free variables, x_3 and x_4, so we may infer that Col $\boldsymbol{A}$ is the span of the independent vectors $\vec{a}_1, \vec{a}_2$ and these also form a basis $\mathcal{B}$ for Col $\boldsymbol{A}$:

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix} \right\}$$

To find a basis for Nul $\boldsymbol{A}$ we first find a description for the vectors in Nul $\boldsymbol{A}$; one such description is the solutions for $\vec{x}$ in $\boldsymbol{A}\vec{x} = \vec{0}$. We can solve this easily by considering $\boldsymbol{B}$ as an echelon form of $\boldsymbol{A}$:

$$\begin{aligned} x_1 + 6t + 5s &= 0 \\ 2x_2 + 5t + 3s &= 0 \\ x_3 &= t \\ x_4 &= s \end{aligned}$$

$$\vec{x} = \begin{bmatrix} -6t - 5s \\ (-5/2)\,t - (3/2)\,s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}$$

The solutions for $\vec{x}$ are the vectors in Nul $\boldsymbol{A}$, so the two vectors in the parametric equation span Nul $\boldsymbol{A}$ and thus form a basis $\mathcal{C}$ for Nul $\boldsymbol{A}$:

$$\mathcal{C} = \left\{ \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

4.3.6 (15–18) Find a basis for the space spanned by the given vectors

15. $\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -6 \\ 9 \end{bmatrix}$

The space V spanned by these vectors is the space spanned by the largest subset of them that is linearly independent. It's not given whether any of the vectors are linearly independent, so we have to identify the spanning set by looking at the reduced echelon form of their matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -3 & 0 & 4 \\ 0 & \boxed{1} & -4 & 0 & -5 \\ 0 & 0 & 0 & \boxed{1} & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are only three pivot columns in the matrix (indicated) so only three of the vectors are linearly independent. The dependent vectors are not part of a basis for V. The space V spanned by these vectors is in fact the column space of $\boldsymbol{A}$ and the independent vectors are a basis $\mathcal{B}$ for it:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix} \right\}$$

☐☐☐ 4.3.7 (19–20) Find a basis for the space spanned by non-independent vectors; how many bases are there? What is the relationship between the vectors' independence and the bases?

19. Let $\vec{v}_1 = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$, and $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.
It can be verified that $4\vec{v}_1 + 5\vec{v}_2 - 3\vec{v}_3 = \vec{0}$. Use this information to find a basis for H. There is more than one answer.

H is the column space of the matrix $\boldsymbol{A} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$. We construct and analyze that matrix:

$$\boldsymbol{A} = \begin{bmatrix} 4 & 1 & 7 \\ -3 & 9 & 11 \\ 7 & -2 & 6 \end{bmatrix}$$

We're given one vector in $\text{Nul}\,\boldsymbol{A}$, $\langle 4, 5, -3 \rangle$, and it is not the zero vector—so we know that not all of these columns are independent and the dimension of H will be lower than 3. Taking the reduced echelon form of $\boldsymbol{A}$ we see there are two pivots

$$\boldsymbol{A} \overset{\text{RREF}}{\longrightarrow} \begin{bmatrix} 1 & 0 & 4/3 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \end{bmatrix}$$

so the dimension of H, and the number of vectors in its bases, is two. Any two of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent, so possible bases for H include $\{\vec{v}_1, \vec{v}_2\}$, $\{\vec{v}_1, \vec{v}_3\}$, $\{\vec{v}_2, \vec{v}_3\}$, among others.

☐☐☐ 4.3.8 (21–22) Support or contradict statements about bases

21. (a) A single vector by itself is linearly dependent.

False; vectors are linearly dependent when their combination has at least one nontrivial solution of coefficients to equal the zero vector. A single vector cannot have nontrivial coefficients to equal the zero vector unless it is the zero vector, in which case all combinations equal the zero vector.

4.3.9 (23) Explain the connection between span, linear independence, and bases

23. Suppose $\mathbb{R}^4 = \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_4\}$. Explain why $\{\vec{v}_1, \ldots, \vec{v}_4\}$ is a basis for $\mathbb{R}^4$.

By the definition of a basis (Lay p. 211), if $\{\vec{v}_1, \ldots, \vec{v}_4\}$ is a linearly independent set and $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_4\}$ coincides with a subspace H then $\{\vec{v}_1, \ldots, \vec{v}_4\}$ forms a basis for H. By properties of invertible matrices, if $\{\vec{v}_1, \ldots, \vec{v}_4\}$ are the columns of a matrix $\boldsymbol{A}$ and the columns of $\boldsymbol{A}$ span $\mathbb{R}^4$ then the columns of $\boldsymbol{A}$ are linearly independent. $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_4\}$ is given to coincide with $\mathbb{R}^4$ so $\{\vec{v}_1, \ldots, \vec{v}_4\}$ must be linearly independent and therefore form a basis for $\mathbb{R}^4$.

Additionally, by the *spanning set theorem* (Lay p. 212) if $S = \{\vec{v}_1, \ldots, \vec{v}_4\}$ and $H = \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_4\}$ then some subset of S is a basis for H.

Finally Theorem $\boxed{6}$ in Lay (p. 214) states that "the pivot columns of a matrix $\boldsymbol{A}$ form a basis for $\operatorname{Col}\boldsymbol{A}$." If $\{\vec{v}_1, \ldots, \vec{v}_4\}$ are the columns of a matrix $\boldsymbol{A}$ then $\boldsymbol{A}$ is an $m \times 4$. The columns of $\boldsymbol{A}$ span $\mathbb{R}^4$ so by properties of invertible matrices $\boldsymbol{A}$ has four pivot columns; by Theorem $\boxed{6}$ then the columns of $\boldsymbol{A}$ form a basis of four vectors. By the definition of the dimension of a vector space (Lay p. 228), $\dim \operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_4\} = \dim \mathbb{R}^4 = 4$, so the columns of $\boldsymbol{A}$ are a basis for $\mathbb{R}^4$.

4.3.10 (33–34) Determine if a set of polynomials is a linearly independent set of vectors; find a basis for them

33. Consider the polynomials $\vec{p}_1(t) = 1 + t^2$ and $\vec{p}_2(t) = 1 - t^2$. Is $\{\vec{p}_1, \vec{p}_2\}$ a linearly independent set in $\mathbb{P}_3$? Why or why not?

The standard basis for $\mathbb{P}_3$ is $\{1, t, t^2\}$ so the standard basis coordinate for $\vec{p}_1, \vec{p}_2$ are

$$\vec{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{p}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

At a glance we can see that these vectors do not have a scalar ratio relating them and so they are linearly independent. We can test this by reducing the matrix $\boldsymbol{A} = \begin{bmatrix}\vec{p}_1 & \vec{p}_2\end{bmatrix}$ to echelon form and observing the pivot columns:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Each vector has a pivot column in $\boldsymbol{A}$ so they are linearly independent.

4.3.11 (37) Prove that a set of functions on $\mathbb{R}$ is linearly independent

37. Show that $\{t, \sin t, \cos 2t, \sin t \cos t\}$ is a linearly independent set of functions defined on $\mathbb{R}$. Start by assuming that

$$c_1 \cdot t + c_2 \cdot \sin t + c_3 \cdot \cos 2t + c_4 \cdot \sin t \cos t = 0$$

This relation must hold for all $t \in \mathbb{R}$, so choose several specific values of t (e.g. $t = 0$, .1, .2) until you get a system of enough equations to determine that all the c_j must be zero.

We let t be each of the values in $\left\{0, \pi, \frac{\pi}{2}, \frac{\pi}{4}\right\}$ and observe the system of equations:

$$\begin{aligned} c_1 \cdot 0 + c_2 \cdot 0 + c_3 \cdot 1 + c_4 \cdot 0 &= 0 \\ c_1 \cdot \pi + c_2 \cdot 0 + c_3 \cdot 1 + c_4 \cdot 0 &= 0 \\ c_1 \cdot \frac{\pi}{2} + c_2 \cdot 1 + c_3 \cdot (-1) + c_4 \cdot 0 &= 0 \\ c_1 \cdot \frac{\pi}{4} + c_2 \cdot \frac{1}{\sqrt{2}} + c_3 \cdot 0 + c_4 \cdot \frac{1}{2} &= 0 \end{aligned}$$

Solving for $\vec{c}$ in $\boldsymbol{A}\vec{c} = \vec{0}$, the augmented matrix of coefficients is

$$\begin{bmatrix} \boldsymbol{A} & \vec{0} \end{bmatrix} = \left[\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ \pi & 0 & 1 & 0 & 0 \\ \pi/2 & 1 & -1 & 0 & 0 \\ \pi/4 & 1/\sqrt{2} & 0 & 1/2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Each variable has a pivot position in $\boldsymbol{A}$ so the only solution for $\vec{c}$ in $\boldsymbol{A}\vec{c} = \vec{0}$ is the trivial one and the functions are linearly independent.

4.4 Coordinate Systems

Exercises to practice

1–17, 27–36

4.4.1 (1–4) Transform the given vector from $\mathcal{B}$ coordinates to standard basis coordinates

1. $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}, [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Construct the change-of-coordinates matrix for $E \leftarrow \mathcal{B}$

$$\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}$$

Transform the vector in $\mathcal{B}$ coordinates by the change-of-coordinates matrix

$$\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}} [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 5(3) + 3(-4) \\ 5(-5) + 3(6) \end{bmatrix} = \begin{bmatrix} 15 - 12 \\ -25 + 18 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

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2. $\mathcal{B} = \left\{ \begin{bmatrix} 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \end{bmatrix} \right\}, [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$

4.4.2 (5–8) Transform the given vector from standard basis coordinates to $\mathcal{B}$ coordinates

Questions to ask

Find the coordinate vector $[\vec{x}]_{\mathcal{B}}$ of $\vec{x}$ relative to the given basis $\mathcal{B} = \left\{ \vec{b}_1, \ldots, \vec{b}_n \right\}$.

5. $\vec{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

The change-of-coordinates matrix from $\mathcal{B}$ to E is

$$\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix}$$

and the change-of-coordinates matrix from E to $\mathcal{B}$ is the inverse of that

$$\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{E}} = \boldsymbol{P}^{-1}_{\mathcal{E}\leftarrow\mathcal{B}} = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -3 & -5 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|cc} 1 & 0 & -5 & -2 \\ 0 & 1 & 3 & 1 \end{array}\right]$$

$$\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{E}} = \begin{bmatrix} -5 & -2 \\ 3 & 1 \end{bmatrix}$$

4.4.3 (9–10) Write the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis (as in 1–4)

9. $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -9 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \end{bmatrix} \right\}$

We construct the change-of-coordinates matrix from $\mathcal{B}$ to E as the matrix that transforms the basis vectors in $\mathcal{B}$ to the standard basis. The vectors given are *in* standard coordinates, so that matrix is simply

$$\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -9 & 8 \end{bmatrix}$$

4.4.4 (11–12) Find the change-of-coordinates matrix and use its inverse to transform the given vector from $\mathcal{B}$ coordinates to standard basis coordinates (as in 5–8)

Use an inverse matrix to find $[\vec{x}]_{\mathcal{B}}$ for the given $\vec{x}$ and $\mathcal{B}$.

11. $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}, \vec{x} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$

Construct the change-of-coordinates matrix from $\mathcal{B}$ coordinates to standard basis coordinates.

$$\begin{aligned}
\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}} &= \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix} \\
\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{E}} &= \boldsymbol{P}^{-1}_{\mathcal{E}\leftarrow\mathcal{B}} \\
&= \left[\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ -5 & 6 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|cc} 1 & 0 & -3 & -2 \\ 0 & 1 & -5/2 & -3/2 \end{array}\right] \\
\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{E}} &= \begin{bmatrix} -3 & -2 \\ -5/2 & -3/2 \end{bmatrix}
\end{aligned}$$

Use the inverse change-of-coordinates matrix to convert $\vec{x}$ from standard basis coordinates to $\mathcal{B}$ coordinates.

$$\begin{aligned}
[\vec{x}]_{\mathcal{B}} = \boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{E}}\,(\vec{x}) &= \begin{bmatrix} -3 & -2 \\ -5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \end{bmatrix} \\
&= \begin{bmatrix} 2(-3) - 6(-2) \\ 2(-5/2) - 6(-3/2) \end{bmatrix} = \begin{bmatrix} -6 + 12 \\ -5 + 9 \end{bmatrix} \\
[\vec{x}]_{\mathcal{B}} &= \begin{bmatrix} 6 \\ 4 \end{bmatrix}
\end{aligned}$$

4.4.5 (13–14) Given a basis $\mathcal{B}$ for $\mathbb{P}_2$ transform the given vector to $\mathcal{B}$ coordinates

13. The set $\mathcal{B} = \{1+t^2, t+t^2, 1+2t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\vec{p}(t) = 1 + 4t + 7t^2$ relative to $\mathcal{B}$.

The vectors in $\mathcal{B}$ can be written in terms of their coordinates on the standard basis E for $\mathbb{P}_2$ ($\{1, t, t^2\}$):

$$\mathcal{B} = \left\{ \left[\vec{b}_1\right]_{\mathcal{E}}, \left[\vec{b}_2\right]_{\mathcal{E}}, \left[\vec{b}_3\right]_{\mathcal{E}} \right\} = \left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix} \right\}$$

The change-of-coordinates matrix from $\mathcal{B}$ to standard basis coordinates comprises these standard coordinates of the vectors in $\mathcal{B}$. Vectors not labeled with a basis are assumed to be in standard basis coordinates:

$$\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 2\\1 & 1 & 1\end{bmatrix}$$

and from the standard basis to $\mathcal{B}$, the change-of-coordinates matrix is

$$\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{E}} = \boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}^{-1} = \begin{bmatrix}1 & 0 & 1\\0 & 1 & 2\\1 & 1 & 1\end{bmatrix}^{-1} = \begin{bmatrix}1/2 & -1/2 & 1/2\\-1 & 0 & 1\\1/2 & 1/2 & -1/2\end{bmatrix}$$

To convert $\vec{p}(t)$ from standard to $\mathcal{B}$ coordinates we multiply:

$$\begin{aligned}[\vec{p}(t)]_{\mathcal{B}} &= \boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{E}}\vec{p}(t)\\ &= \begin{bmatrix}1/2 & -1/2 & 1/2\\-1 & 0 & 1\\1/2 & 1/2 & -1/2\end{bmatrix}\begin{bmatrix}1\\4\\7\end{bmatrix}\\ &= \begin{bmatrix}1/2 - 4/2 + 7/2\\-1+7\\1/2 + 4/2 - 7/2\end{bmatrix} = \begin{bmatrix}2\\6\\-1\end{bmatrix}\end{aligned}$$

thus $\vec{p}(t)$ in $\mathcal{B}$ coordinates is $[\vec{p}(t)]_{\mathcal{B}} = 2 + 6t - t^2$.

4.4.6 (15–16) Support or contradict statements about a vector space V, a basis $\mathcal{B}$ for V, and the change-of-coordinates matrix $\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}$

15. (a) If $\vec{x}$ is in V and if $\mathcal{B}$ contains n vectors, then the $\mathcal{B}$-coordinate vector of $\vec{x}$ is in $\mathbb{R}^n$.

True; If $\mathcal{B}$ contains n vectors then $[\vec{x}]_{\mathcal{B}}$ has n entries and so is in $\mathbb{R}^n$.

(b) If $\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ is the change-of-coordinates matrix, then $[\vec{x}]_{\mathcal{B}} = \boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}\vec{x}$ for $\vec{x}$ in V.

False; $\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ is the change-of-coordinates matrix from $\mathcal{B}$ coordinates to standard basis coordinates. $\vec{x}$ is a standard basis coordinate vector, so the matrix that changes $\vec{x}$ to $[\vec{x}]_{\mathcal{B}}$ is $\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{E}} = \boldsymbol{P}^{-1}_{\mathcal{E}\leftarrow\mathcal{B}}$.

(c) The vector spaces $\mathbb{P}_3$ and $\mathbb{R}^3$ are isomorphic.

False; the vector space $\mathbb{P}_3$ contains polynomials of degree 3 which have the form $c_1 + c_2x + c_3x^2 + c_4x^3$. Coordinate vectors in $\mathbb{P}_3$ have four entries (c_1, c_2, c_3, c_4) while coordinate vectors in $\mathbb{R}^3$ have three entries.

4.4.7 (17) Demonstrate how vectors that do not form a basis are linearly dependent by writing them as combinations of each other

17. The vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$ span $\mathbb{R}^2$ but do not form a basis. Find two different ways to express $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

We find a description of all the vectors $\vec{x}$ of weights such that $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \langle 1, 1\rangle$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -3 \\ -3 & -8 & 7 \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{A} & \langle 1,1\rangle \end{bmatrix} \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -5 & 5 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$\begin{aligned} x_1 &= 5 + 5t \\ x_2 &= -2 - t \\ x_3 &= t \end{aligned}$$

$$\vec{x} = t\begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$$

We've now described "$\langle 1, 1\rangle$" space for $\boldsymbol{A}$; choosing the coordinate t gives us a unique $\vec{x}$ such that $\boldsymbol{A}\vec{x} = \langle 1, 1\rangle$:

$$\text{let } t = 1, \quad \vec{x} = \begin{bmatrix} 10 \\ -3 \\ 1 \end{bmatrix}, \quad \boldsymbol{A}\vec{x} = \begin{bmatrix} 10 - 6 - 3 \\ -30 + 24 + 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{let } t = 2, \quad \vec{x} = \begin{bmatrix} 15 \\ -4 \\ 2 \end{bmatrix}, \quad \boldsymbol{A}\vec{x} = \begin{bmatrix} 15 - 8 - 6 \\ -45 + 32 + 14 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

☐☐☐ **4.4.8 (27–31) Write the given polynomials as coordinate vectors in $\mathbb{P}_n$ and test their linear independence**

27. $1+2t^3$, $2+t-3t^2$, $-t+2t^2-t^3$

Being third degree polynomials these are vectors in $\mathbb{P}_3$ where the standard basis is $\{1,t,t^2,t^3\}$. Their respective coordinate vectors then are

$$\begin{bmatrix}1\\0\\0\\2\end{bmatrix}, \begin{bmatrix}2\\1\\-3\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\2\\-1\end{bmatrix}$$

Test linear independence by constructing a matrix with columns of these vectors and put it in reduced echelon form:

$$\begin{bmatrix}1&2&0\\0&1&-1\\0&-3&2\\2&0&-1\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

We have three pivots for three vectors so they are linearly independent and form a basis in $\mathbb{P}_3$.

☐☐☐ **4.4.9 (32) Use coordinate vectors (as in 27–31) to show that the given polynomials form a basis for $\mathbb{P}_2$; find a change-of-coordinates matrix for that basis and apply it to the given vector**

32. Let $\vec{p}_1(t)=1+t^2$, $\vec{p}_2(t)=t-3t^2$, $\vec{p}_3(t)=1+t-3t^2$.

(a) Use coordinate vectors to show that these polynomials form a basis for $\mathbb{P}_2$.

The standard basis for $\mathbb{P}_2$ is $\mathcal{E}=\{1,t,t^2\}$; the standard basis coordinates of $\vec{p}_1,\vec{p}_2,\vec{p}_3$ are then

$$\vec{p}_1=\begin{bmatrix}1\\0\\1\end{bmatrix}, \vec{p}_2=\begin{bmatrix}0\\1\\-3\end{bmatrix}, \vec{p}_3=\begin{bmatrix}1\\1\\-3\end{bmatrix}$$

We test linear independence by constructing a matrix $\begin{bmatrix}\vec{p}_1 & \vec{p}_2 & \vec{p}_3\end{bmatrix}$ and taking its reduced echelon form:

$$\begin{bmatrix}1&0&1\\0&1&1\\1&-3&-3\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

The matrix has three pivot columns so $\{\vec{p}_1,\vec{p}_2,\vec{p}_3\}$ is a linearly independent set.

(b) Consider the basis $\mathcal{B} = \{\vec{p}_1, \vec{p}_2, \vec{p}_3\}$ for $\mathbb{P}_2$. Find $\vec{q}$ in $\mathbb{P}_2$, given that $[\vec{q}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

Construct the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis $\mathcal{E}$ as $\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} \vec{p}_1 & \vec{p}_2 & \vec{p}_3 \end{bmatrix}$ and multiply by $[\vec{q}]_{\mathcal{B}}$ to obtain $\vec{q}$:

$$\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -3 & -3 \end{bmatrix}$$

$$\vec{q} = \boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}\,[\vec{q}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -3 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -10 \end{bmatrix}$$

☐☐☐ **4.4.10 (33–34) Use a calculator to test if the given polynomials form a basis for $\mathbb{P}_3$; explain how this is determined**

33. $3 + 7t$, $5 + t - 2t^3$, $t - 2t^2$, $1 + 16t - 6t^2 + 2t^3$

The standard basis for $\mathbb{P}_3$ is $\{1, t, t^2, t^3\}$ and the matrix of standard basis coordinates of these vectors is

$$\boldsymbol{A} = \begin{bmatrix} 3 & 5 & 0 & 1 \\ 7 & 1 & 1 & 16 \\ 0 & 0 & -2 & -6 \\ 0 & -2 & 0 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The standard basis for $\mathbb{P}_3$ has four vectors so by the definition of dimension $\dim \mathbb{P}_3 = 4$ and all bases for $\mathbb{P}_3$ have four vectors. Every basis consists of linearly independent vectors; there are not four linearly independent vectors in the given set so they cannot span $\mathbb{P}_3$ and cannot form a basis for it.

☐☐☐ **4.4.11 (35–36) Construct a subspace H from the given basis vectors $\vec{v}_n$ and transform the given vector $\vec{x}$ to that basis; how do you test if $\vec{x}$ is in H?**

35. Let $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ and $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$. Show that $\vec{x}$ is in H and find the $\mathcal{B}$-coordinate vector of $\vec{x}$, for

$$\vec{v}_1 = \begin{bmatrix} 11 \\ -5 \\ 10 \\ 7 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 14 \\ -8 \\ 13 \\ 10 \end{bmatrix}, \vec{x} = \begin{bmatrix} 19 \\ -13 \\ 18 \\ 15 \end{bmatrix}$$

The dimension of H is the number of vectors in its bases; $\{\vec{v}_1, \vec{v}_2\}$ span H and so form a basis for it, therefore $\dim H = 2$. Let $\boldsymbol{A}$ be the matrix $\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}$. If $\vec{x} \in H$ then there exists a solution for $\vec{y}$ in $\boldsymbol{A}\vec{y} = \vec{x}$. Augment $\begin{bmatrix}\boldsymbol{A} & \vec{x}\end{bmatrix}$ and row reduce to find the solution for $\vec{y}$, if it exists:

$$\begin{bmatrix}\boldsymbol{A} & \vec{x}\end{bmatrix} = \left[\begin{array}{cc|c} 11 & 14 & 19 \\ -5 & -8 & -13 \\ 10 & 13 & 18 \\ 7 & 10 & 15 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & -5/3 \\ 0 & 1 & 8/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Every vector in $\boldsymbol{A}$ has a pivot in the reduced echelon form so the solution for $\vec{y}$ exists and is $\begin{bmatrix} -5/3 \\ 8/3 \end{bmatrix}$. Now, $\boldsymbol{A}$ is identical to the change-of-coordinates matrix $\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}$. Changing a vector from $\mathcal{B}$-coordinates to standard basis coordinates takes the form $\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}\,[\vec{v}]_{\mathcal{B}} = \vec{v}$, which is identical to the above form $\boldsymbol{A}\vec{y} = \vec{x}$. It follows then that the solution for $\vec{y}$ is the $\mathcal{B}$-coordinate vector $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -5/3 \\ 8/3 \end{bmatrix}$.

4.5 The Dimension of a Vector Space

Exercises to practice

1–30, 33

Questions to ask

4.5.1 (1–8) Find a basis for the given subspace and state the dimension of the subspace; understand how dimension of a subspace is related to the dimension of its bases

For each subspace, (a) find a basis and (b) state the dimension.

1. $H = \left\{ \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} : s,t \in \mathbb{R}^3 \right\}$

(a) A basis for H is a minimal set of vectors that spans it. From properties of invertible matrices we can deduce that any linearly independent set of vectors in a vector space V will span V and thus will be a basis for V. Also from properties of invertible matrices we know that a linearly independent set of vectors in an n-dimensional vector space will have n pivots and therefore must comprise n vectors. Additionally, from the *basis theorem* ($\boxed{12}$ in Lay p. 229) we know that n linearly independent vectors span an n-dimensional vector space and form a basis for that space.

A set of basis vectors is immediately apparent when we write H in parametric form. Consider that s and t are coordinates of vectors in H and look at how linear combinations of those coordinates form vectors in H

$$\begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

Clearly a basis for H is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$. The dimension of H is defined as the number of vectors in a basis for H, so $\dim H = 2$.

4.5.2 (9) Construct a subspace from the given constraint on $\mathbb{R}^2$ and find its dimension; understand what kind of subspace the constraints describe and how its bases relate to its dimension

9. Find the dimension of the subspace of all vectors in $\mathbb{R}^3$ whose first and third entries are equal.

This is the set

$$H = \left\{ \begin{bmatrix} s \\ t \\ s \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

Its members can be expressed by the parametric form

$$s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

in which $\langle s, t \rangle$ constitute coordinates in a 2-dimensional subspace of $\mathbb{R}^3$. Thus the set H is a subspace of $\mathbb{R}^3$ that is isomorphic to $\mathbb{R}^2$.

4.5.3 (10–12) Given a set of vectors, find the dimension of the subspace they span

10. Find the dimension of the subspace H of $\mathbb{R}^2$ spanned by

$$\begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}$$

$\mathbb{R}^2$ is completely spanned by two independent vectors, so one of these vectors cannot be independent from the others. Whether there is *more than one* dependent vector is not obvious, however; the dimension of H may be 2 or 1.

By properties of invertible matrices we can say that any independent set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ will have only the trivial solution to $\boldsymbol{A}\vec{x} = \vec{0}$ where $\boldsymbol{A} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$. We augment $\begin{bmatrix} \boldsymbol{A} & \vec{0} \end{bmatrix}$ and take the reduced echelon form to identify solutions to the equation:

$$\left[\begin{array}{ccc|c} 2 & -4 & -3 & 0 \\ -5 & 10 & 6 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Thus the first and second vectors are linearly dependent while the third is independent from both. The two independent vectors therefore comprise a basis for H and so H has a dimension of 2.

4.5.4 (13–18) Given a matrix $\boldsymbol{A}$ determine the dimensions of $\operatorname{Nul}\boldsymbol{A}$ and $\operatorname{Col}\boldsymbol{A}$

13. $\boldsymbol{A} = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Find $\dim \operatorname{Col} \boldsymbol{A}$ by identifying the independent columns of $\boldsymbol{A}$:

$$\begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 21 & 0 & 164/5 \\ 0 & \boxed{1} & 2 & 0 & 29/5 \\ 0 & 0 & 0 & \boxed{1} & 1/5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot positions are indicated with boxes; here we have three independent columns so $\dim \operatorname{Col} \boldsymbol{A} = 3$.

Now the *rank theorem* ($\boxed{14}$ in Lay p. 235) tells us that for a $m \times n$ matrix,

$$\begin{aligned} \dim \operatorname{Nul} \boldsymbol{A} &= n - \dim \operatorname{Col} \boldsymbol{A} = 5 - 3 \\ &= 2 \end{aligned}$$

Alternatively, find $\operatorname{Nul} \boldsymbol{A}$ by writing the above columns in parametric form:

$$\begin{aligned} x_3 &= t \\ x_5 &= s \\ x_1 &= -21t - \frac{164}{5}s \\ x_2 &= -2t - \frac{29}{5}s \\ x_4 &= -\frac{1}{5}s \end{aligned}$$

$$\operatorname{Nul} \boldsymbol{A} = \left\{ \begin{bmatrix} -21t - (164/5)\, s \\ -2t - (29/5)\, s \\ t \\ (-1/5)\, s \\ s \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} -21 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 164/5 \\ 29/5 \\ 0 \\ -1/5 \\ 1 \end{bmatrix} \right\}$$

The dimension of $\operatorname{Nul} \boldsymbol{A}$ is 2.

4.5.5 (19–20) Support or contradict statements about the dimensions of vector spaces

V is a vector space. Mark each statement *True* or *False* and justify your answer.

19. (a) The number of pivot columns of a matrix equals the dimension of its column space.

True; the pivot columns are the independent column vectors in the matrix and are sufficient to span the column space. By Theorem $\boxed{12}$ in Lay (p. 229), "Any set of exactly p elements that spans V is automatically a basis for V." Here we have exactly p pivot columns that span the column space, so they are a basis for it. By definition the dimension of the column space is the number of vectors in its bases (Lay p. 228), so the dimension of the column space of this matrix is the number of pivot columns.

Lay summarizes this on p. 230:

> ... the dimension of $\operatorname{Col} \boldsymbol{A}$ is the number of pivot columns in $\boldsymbol{A}$.

(b) A plane in $\mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$.

True; the plane is a subspace of $\mathbb{R}^3$ because the vectors on it have three entries, but it is a two-dimensional subspace because exactly two vectors are necessary to establish a basis for it.

(c) The dimension of the vector space $\mathbb{P}_4$ is 4.

False; the vector space $\mathbb{P}_4$ contains polynomials of degree 4. Fourth degree polynomials take the form $1+x+x^2+x^3+x^4$, so there are actually *five* entries in their coordinate vectors. A basis for $\mathbb{P}_4$ would therefore require exactly five vectors to span $\mathbb{P}_4$, so $\dim \mathbb{P}_4 = 5$.

(d) If $\dim V = n$ and S is a linearly independent set in V, then S is a basis for V.

False; S being a linearly independent set is necessary, but not sufficient, for S to be a basis for V; S must also contain exactly n elements, because $\dim V = n$ and $\dim V$ is by definition the number of elements in the bases for V (Theorem $\boxed{10}$ and the definition in Lay p. 228).

(e) If a set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ spans a finite-dimensional vector space V and if T is a set of more than p vectors in V, then T is linearly dependent.

True; if the set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ spans V (and there are no more elements in the set than necessary) then the elements of the set are a basis for V and $\dim V = p$ (definition in Lay p. 228 and Theorem $\boxed{12}$ p. 229). If T is a set containing more than p vectors in V then T must contain linearly dependent vectors (Theorem $\boxed{9}$ in Lay p. 227).

☐☐☐ 4.5.6 (21–22) Prove that the given polynomials are a basis for $\mathbb{P}_3$

21. The first four Hermite polynomials are 1, $2t$, $-2+4t^2$, and $-12t+8t^3$. These polynomials arise naturally in the study of certain important differential equations in mathematical physics. Show that the first four Hermite polynomials form a basis for $\mathbb{P}_3$.

As coordinate vectors, these are

$$\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\4\\0\end{bmatrix}, \begin{bmatrix}0\\-12\\0\\8\end{bmatrix} \right\}$$

They are linearly independent,

$$\begin{bmatrix}1&0&-2&0\\0&2&0&-12\\0&0&4&0\\0&0&0&8\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

so by the *basis theorem* they form a basis for $\mathbb{P}_3$:

> Let V be a p-dimensional vector space. Any linearly independent set of exactly p elements in V is automatically a basis for V. (Theorem 12 in Lay p. 229)

4.5.7 (23–24) Use a change of coordinates matrix to find the given vector relative to the given basis

23. Let $\mathcal{B}$ be the basis of $\mathbb{P}_3$ consisting of the Hermite polynomials in Exercise 21 on the preceding page, and let $\vec{p}(t) = 7 - 12t - 8t^2 + 12t^3$. Find the coordinate vector of $\vec{p}$ relative to $\mathcal{B}$.

We're given the vectors of $\mathcal{B}$ in standard basis coordinates so we construct the change-of-coordinates matrix $\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ from those vectors. We're asked to convert *to* $\mathcal{B}$-coordinates, however, so we invert $\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ to obtain $\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{E}}$. Borrowing the coordinate vectors for $\mathcal{B}$ from Exercise 21, we have

$$\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{E}} = \boldsymbol{P}^{-1}_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix}1&0&-2&0\\0&2&0&-12\\0&0&4&0\\0&0&0&8\end{bmatrix}^{-1} = \begin{bmatrix}1&0&1/2&0\\0&1/2&0&3/4\\0&0&1/4&0\\0&0&0&1/8\end{bmatrix}$$

And the standard basis coordinates of $\vec{p}$ are

$$[\vec{p}]_{\mathcal{E}} = \begin{bmatrix}7\\-12\\-8\\12\end{bmatrix}$$

so the conversion is

$$[\vec{p}]_{\mathcal{B}} = \boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{E}}\,[\vec{p}]_{\mathcal{E}} = \begin{bmatrix}1&0&1/2&0\\0&1/2&0&3/4\\0&0&1/4&0\\0&0&0&1/8\end{bmatrix}\begin{bmatrix}7\\-12\\-8\\12\end{bmatrix} = \begin{bmatrix}3\\3\\-2\\3/2\end{bmatrix}$$

4.5.8 (25–28) Understand why a basis must have enough vectors to span the subspace

25. Let S be a subset of an n-dimensional vector space V, and suppose S contains fewer than n vectors. Explain why S cannot span V.

If S spanned V then it would contain enough linearly independent vectors to form a basis for V. The definition of the dimension of V is the exact number of vectors of all of its bases. If V is n-dimensional and S contains a basis for V then, by the definition of dimension, S contains at least n linearly independent vectors. Because S is stated to contain fewer than n vectors, it cannot contain a basis for V. Every set of vectors spanning a vector space is a basis for that space, so if S does not contain a basis for V then it does not contain a set of vectors that span V.

We can prove that S does not span V by contradiction: suppose that S does span V. By the *spanning set theorem* ($\boxed{5}$ in Lay p. 212), some subset of S is a basis $\mathcal{B}$ for V. Because S contains m vectors such that $m < n$, any basis for V that is a subset of S has at most m vectors. By the definition of dimension (Lay p. 228) the dimension of V is the number of vectors in $\mathcal{B}$, which is at most m. The dimension of V has been given as n and $m < n$ so $\mathcal{B}$ cannot be a basis for V, so there is no subset of S which spans V.

4.5.9 (29–30) Support or contradict statements about the span and independence of vectors in a subspace; understand the relationship between the dimension of a subspace and its bases

29. V is a nonzero finite-dimensional vector space, and the vectors listed belong to V. Mark each statement *True* or *False* and justify your answer.

(a) If there exists a set $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ that spans V then $\dim V \leq p$.

True; if S spans V then a subset of S is linearly independent and spans V. By the *spanning set theorem* (Lay p. 212) if any of the vectors in S are not linearly independent then the subset of S not containing those vectors still spans V. By definition the dimension of V is the number of vectors that form a basis for V (Lay p. 228). By the *basis theorem* ($\boxed{12}$ in Lay p. 229) any set of p elements that spans V is a basis for V. S spans V so the minimal subset of S that still spans V is a basis for V and that subset has p or fewer elements; the dimension of V is therefore at most p.

(b) If there exists a linearly independent set $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ in V then $\dim V \geq p$.

True; any basis of V is a linearly independent set of k elements where $k = \dim V$. If a set S of p elements in V is linearly independent then S either forms a basis for V (and $k = p$) or is a subset of a basis for V (and $k > p$). The dimension of V then is either $k = p$ or $k > p$, so $\dim V \geq p$.

(c) If $\dim V = p$, then there exists a spanning set of $p + 1$ vectors in V.

True; if a set of $p+1$ vectors span V then one of those vectors is dependent and is not included in the minimal subset that still spans V.

4.5.10 (33) Use the given independent vectors plus enough standard basis vectors to form a basis for a higher dimensional vector space

33. According to Theorem 11 (Lay p. 229) a linearly independent set $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$ can be expanded to a basis for $\mathbb{R}^n$. One way to do this is to create $\boldsymbol{A} = [\vec{v}_1 \ \cdots \ \vec{v}_k \ \vec{e}_1 \ \cdots \ \vec{e}_n]$, with $\vec{e}_1, \ldots, \vec{e}_n$ the columns of the identity matrix (standard basis vectors); the pivot columns of $\boldsymbol{A}$ form a basis for $\mathbb{R}^n$.

(a) Use the method described to extend the following vectors to a basis for $\mathbb{R}^5$:

$$\vec{v}_1 = \begin{bmatrix} -9 \\ -7 \\ 8 \\ -5 \\ 7 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 9 \\ 4 \\ 1 \\ 6 \\ -7 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 6 \\ 7 \\ -8 \\ 5 \\ -7 \end{bmatrix}$$

We're given a set of three independent vectors in $\mathbb{R}^5$. We know that the dimension of any basis in $\mathbb{R}^n$ will be n; having three vectors we need two more that are independent of each other and of the three given. Two vectors that we know are independent of each other are any two of the standard basis vectors in $\mathbb{R}^5$.

Because the given vectors $\vec{v}_i$ are independent of each other and do not span $\mathbb{R}^5$, we know that they cannot be combinations of all of the standard basis vectors in $\mathbb{R}^5$—if they were then they would span $\mathbb{R}^5$. There are three independent vectors $\vec{v}_i$, so the largest subspace for which they can be a basis has dim 3; and the greatest number of standard basis vectors they combine is three. Thus if we solve which of the standard basis vectors $\{\vec{e}_a, \vec{e}_b, \vec{e}_c\}$ combine to give $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ then we can choose the remaining vectors $\{\vec{e}_d, \vec{e}_e\}$ to complete the basis.

We begin by augmenting the three given vectors with all of the standard basis vectors in $\mathbb{R}^5$:

$$\boldsymbol{A} = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \ | \ \vec{e}_1 \ \cdots \ \vec{e}_5]$$

By reducing $\boldsymbol{A}$ to echelon form we find the pivots of all of the vectors it contains. Any vector that has a pivot position is linearly independent of some of the other vectors; any vector without a pivot is independent of none of them. Based on statements in the previous paragraph we predict that we will find exactly two of the standard basis vectors to have pivot positions and that they will both be independent of the given vectors $\vec{v}_i$.

$$\left[\begin{array}{rrr|rrrrr} -9 & 9 & 6 & 1 & 0 & 0 & 0 & 0 \\ -7 & 4 & 7 & 0 & 1 & 0 & 0 & 0 \\ 8 & 1 & -8 & 0 & 0 & 1 & 0 & 0 \\ -5 & 6 & 5 & 0 & 0 & 0 & 1 & 0 \\ 7 & -7 & -7 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{RREF} \left[\begin{array}{rrr|rrrrr} 1 & 0 & 0 & -1/3 & 0 & 0 & 1 & 3/7 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 5/7 \\ 0 & 0 & 1 & -1/3 & 0 & 0 & 0 & -3/7 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 3 & 22/7 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & -9 & -53/7 \end{array}\right]$$

Thus the two additional vectors which are independent of each other and of $\vec{v}_i$ are $\vec{e}_2$ and $\vec{e}_3$, so a basis for $\mathbb{R}^5$ is

$$\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{e}_2, \vec{e}_3\}$$

(b) Explain why the method works in general: Why are the original vectors $\vec{v}_i$ included in the basis found for Col $\boldsymbol{A}$? Why is Col $\boldsymbol{A} = \mathbb{R}^n$?

4.6 Rank

Exercises to practice

1–18

All of the questions in this section aim to exercise one skill: understand how the rank, nullity, column space, row space, and bases cooperate to describe a matrix. **None of these properties is independent of all the others**, and the questions here force you to consider a matrix in terms of each combination of its properties so that you engrain all the dependencies between them.

4.6.1 (1–4) Determine $\operatorname{rank} \boldsymbol{A}$ and $\dim \operatorname{Nul} \boldsymbol{A}$ by looking at a matrix $\boldsymbol{A}$; then find bases for $\operatorname{Col} \boldsymbol{A}$, $\operatorname{Row} \boldsymbol{A}$, and $\operatorname{Nul} \boldsymbol{A}$

Assume the matrix $\boldsymbol{A}$ is row equivalent to $\boldsymbol{B}$. Without calculations, list $\operatorname{rank} \boldsymbol{A}$ and $\dim \operatorname{Nul} \boldsymbol{A}$. Then find bases for $\operatorname{Col} \boldsymbol{A}$, $\operatorname{Row} \boldsymbol{A}$, and $\operatorname{Nul} \boldsymbol{A}$.

1. $A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Row equivalence between $\boldsymbol{A}$ and $\boldsymbol{B}$ gives us that:

because $\boldsymbol{B}$ has two free variables, $\boldsymbol{A}$ must also have two free variables

(the first free variable x_3 is so because there is no pivot in the third row so it must be dependent; the second, x_4, is free because there are more vectors than rows—so the fourth vector must be dependent because there is no degree of freedom for it to independently span).

Questions to ask

The span of the vectors in $\boldsymbol{A}$, called $\operatorname{Col} \boldsymbol{A}$, has a dimension of 2; that is, the number of basis vectors needed to span the same space is 2. The rank of $\boldsymbol{A}$ is therefore $\operatorname{rank} \boldsymbol{A} = \dim \operatorname{Col} \boldsymbol{A} = 2$.

The nullity of $\boldsymbol{A}$ ($\dim \operatorname{Nul} \boldsymbol{A}$) is the dimension of the vectors that are mapped to the zero vector under transformation by $\boldsymbol{A}$. Because there are two free variables in $\boldsymbol{A}$, there are infinitely many vectors which are mapped to $\vec{0}$ as we vary either of those free variables; those are the vectors in $\operatorname{Nul} \boldsymbol{A}$. The dimension of that space of vectors, $\dim \operatorname{Nul} \boldsymbol{A}$, is the nullity of $\boldsymbol{A}$. In this case $\dim \operatorname{Nul} \boldsymbol{A} = 2$.

By the *rank theorem* (14 in Lay p. 235) we know that when $\boldsymbol{A}$ is an $m \times n$ matrix, $\operatorname{rank} \boldsymbol{A} + \dim \operatorname{Nul} \boldsymbol{A} = n$. In this case $\boldsymbol{A}$ is a 3×4, and $\operatorname{rank} \boldsymbol{A} + \dim \operatorname{Nul} \boldsymbol{A} = 2 + 2 = 4 = n$.

4.6.2 (5–6) Given $\operatorname{rank} \boldsymbol{A}$ find $\dim \operatorname{Nul} \boldsymbol{A}$, $\dim \operatorname{Row} \boldsymbol{A}$, and $\operatorname{rank} \boldsymbol{A}^T$

5. A 3×8 matrix $\boldsymbol{A}$ with rank 3

By the *rank theorem* (14 in Lay p. 235), $\operatorname{rank} \boldsymbol{A} + \dim \operatorname{Nul} \boldsymbol{A} = n$; so $3 + \dim \operatorname{Nul} \boldsymbol{A} = 8$ and $\dim \operatorname{Nul} \boldsymbol{A} = 5$. The row space of $\boldsymbol{A}$ is the column space of $\boldsymbol{A}^T$, so $\operatorname{Row} \boldsymbol{A} = \operatorname{Col} \boldsymbol{A}^T$ and $\dim \operatorname{Row} \boldsymbol{A} = \dim \operatorname{Col} \boldsymbol{A}^T = \dim \operatorname{Col} \boldsymbol{A} = 3$ (transposing $\boldsymbol{A}$ does not change the number of pivots). The rank of $\boldsymbol{A}^T$ is defined as $\dim \operatorname{Col} \boldsymbol{A}^T$, so $\operatorname{rank} \boldsymbol{A}^T = 3$.

4.6.3 (7–8) Describe the column space and nullity of $\boldsymbol{A}$ given the number of pivots in $\boldsymbol{A}$

7. Suppose a 4×7 matrix $\boldsymbol{A}$ has four pivot columns. Is $\operatorname{Col} \boldsymbol{A} = \mathbb{R}^4$? Is $\operatorname{Nul} \boldsymbol{A} = \mathbb{R}^3$? Explain your answers.

Because $\boldsymbol{A}$ has four pivot columns (linearly independent columns), the span of those columns $\operatorname{Col} \boldsymbol{A}$ has four dimensions. Further, the vectors in $\operatorname{Col} \boldsymbol{A}$ have four dimensions because $\boldsymbol{A}$ has four rows. The span of $\operatorname{Col} \boldsymbol{A}$ is a four-dimensional subset of $\mathbb{R}^4$ and the only four dimensional subset of $\mathbb{R}^4$ is identical to $\mathbb{R}^4$, so $\operatorname{Col} \boldsymbol{A} = \mathbb{R}^4$.

By the *rank theorem* (14 in Lay p. 235), the dimension of $\operatorname{Nul} \boldsymbol{A}$ is $n - \dim \operatorname{Col} \boldsymbol{A} = 7 - 4 = 3$. Vectors in $\operatorname{Nul} \boldsymbol{A}$ have seven entries and so can't be vectors in $\mathbb{R}^3$; rather $\operatorname{Nul} \boldsymbol{A}$ is a subspace of $\mathbb{R}^7$ and it has three dimensions (by the rank theorem, there are three basis vectors for $\operatorname{Nul} \boldsymbol{A}$).

4.6.4 (9–12) Describe the column and row spaces of $\boldsymbol{A}$ given a description of its null space

9. If the null space of a 5×6 matrix $\boldsymbol{A}$ is 4-dimensional, what is the dimension of the column space of $\boldsymbol{A}$?

By the *rank theorem* (14 in Lay p. 235), $\dim \operatorname{Col} \boldsymbol{A} = \operatorname{rank} \boldsymbol{A} - \dim \operatorname{Nul} \boldsymbol{A} = 6 - 4 = 2$.

4.6.5 (13–16) Describe what possible dimensions the given matrix can have

13. If $\boldsymbol{A}$ is a 7×5 matrix, what is the largest possible rank of $\boldsymbol{A}$? If $\boldsymbol{A}$ is a 5×7?

If $\boldsymbol{A}$ is a 7×5 then it has 5 columns; if all of them are linearly independent then it's rank is $\operatorname{rank} \boldsymbol{A} = \dim \operatorname{Col} \boldsymbol{A} = 5$.

If $\boldsymbol{A}$ is a 5×7 then it's largest possible rank is still 5 because it can't have more than 5 linearly independent columns (it can't have more than 5 pivots).

4.6.6 (17–18) Support or contradict statements about the dimensions of the given matrix and its bases

17. $\boldsymbol{A}$ is an $m \times n$ matrix. Mark each statement *True* or *False* and justify your answers.

(a) The row space of $\boldsymbol{A}$ is the same as the column space of $\boldsymbol{A}^T$.

True; the rows of $\boldsymbol{A}$ are identical to the columns of $\boldsymbol{A}^T$.

(b) If $\boldsymbol{B}$ is any echelon form of $\boldsymbol{A}$, and if $\boldsymbol{B}$ has three nonzero rows, then the first three rows of $\boldsymbol{A}$ form a basis for Row $\boldsymbol{A}$.

False; the rows which form a basis are the nonzero rows of $\boldsymbol{B}$. The first three rows are not necessarily the rows which in $\boldsymbol{B}$ are nonzero, and they may be linearly dependent. The nonzero rows in the echelon form $\boldsymbol{B}$ are linearly independent, however, and may form a basis of a three-dimensional vector space.

(c) The dimensions of the row space and the column space of $\boldsymbol{A}$ are the same, even if $\boldsymbol{A}$ is not square.

True; by the rank theorem (14 in Lay p. 235), this dimension is called the *rank* of $\boldsymbol{A}$. The number of pivots in $\boldsymbol{A}$ is equal to the number of pivots in $\boldsymbol{A}^T$ and the rows of $\boldsymbol{A}$ are the columns of $\boldsymbol{A}^T$; so the dimension of $\operatorname{Col} \boldsymbol{A}^T$ is equal to the dimension of $\operatorname{Col} \boldsymbol{A}$, and the dimension of $\operatorname{Row} \boldsymbol{A}$ is $\operatorname{Col} \boldsymbol{A}^T$.

(d) The sum of the dimensions of the row space and the null space of $\boldsymbol{A}$ equals the number of rows in $\boldsymbol{A}$.

False; the rank theorem states that the sum of the dimensions of the column space (the *rank* of $\boldsymbol{A}$) and null space of an $m \times n$ is n. $\dim \operatorname{Row} \boldsymbol{A} = \dim \operatorname{Col} \boldsymbol{A}$ (see part [c]), so the correct statement would be that the sum of the dimensions of the row space and null space of $\boldsymbol{A}$ equals the number of *columns* in $\boldsymbol{A}$.

4.7 Change of Basis

Exercises to practice

1–14, 19

4.7.1 (1–2, 5–6) Find the change-of-coordinates matrix for the given bases of a vector space; use the matrix to change the coordinates of the given vector

2. Let $\mathcal{B} = \left\{\vec{b}_1, \vec{b}_2\right\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be bases for a vector space V, and suppose $\vec{b}_1 = -\vec{c}_1 + 4\vec{c}_2$ and $\vec{b}_2 = 5\vec{c}_1 - 3\vec{c}_2$.

(a) Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

A change-of-coordinates from $\mathcal{B}$ to $\mathcal{C}$ is equivalent to changing from $\mathcal{B}$ to standard basis E and from E to $\mathcal{C}$. The goal is to construct a matrix that represents $\mathcal{B}$ coordinates in $\mathcal{C}$ coordinates; we do this by converting $\mathcal{B}$ coordinates to standard coordinates and then standard coordinates to $\mathcal{C}$ coordinates.

The given basis vectors $\vec{b}_1$ and $\vec{b}_2$ are *already* defined in terms of the $\mathcal{C}$ basis vectors $\vec{c}_1$ and $\vec{c}_2$—the $\mathcal{B}$ vectors are defined in $\mathcal{C}$ coordinates. All we do then is construct the matrix that transforms those $\mathcal{B}$ vectors to the $\mathcal{C}$ vectors. Applying that matrix to transform any vector in $\mathcal{B}$ will similarly transform that vector to its coordinates relative to the $\mathcal{C}$ vectors.

First we write out the $\mathcal{B}$ vectors as coordinates on the $\mathcal{C}$ basis, to make the relationship clear

Questions to ask

$$\vec{b}_1 = x_1\vec{c}_1 + y_1\vec{c}_2 = (-1)\cdot\vec{c}_1 + 4\cdot\vec{c}_2$$
$$\vec{b}_2 = x_2\vec{c}_1 + y_2\vec{c}_2 = 5\cdot\vec{c}_1 + (-3)\cdot\vec{c}_2$$

$$\begin{bmatrix}\vec{b}_1 & \vec{b}_2\end{bmatrix} = \begin{bmatrix}\vec{c}_1 & \vec{c}_2\end{bmatrix}\begin{bmatrix}x_1 & x_2\\ y_1 & y_2\end{bmatrix} = \begin{bmatrix}-1 & 4\\ 5 & -3\end{bmatrix} = \begin{bmatrix}\vec{c}_1 & \vec{c}_2\end{bmatrix}\begin{bmatrix}\left[\vec{b}_1\right]_{\mathcal{C}} & \left[\vec{b}_2\right]_{\mathcal{C}}\end{bmatrix}$$

The change-of-coordinates matrix is simply the $\mathcal{C}$ coordinates $\langle x_i, y_i\rangle$ of the $\mathcal{B}$ vectors

$$\boldsymbol{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}\left[\vec{b}_1\right]_{\mathcal{C}} & \left[\vec{b}_2\right]_{\mathcal{C}}\end{bmatrix} = \begin{bmatrix}x_1 & x_2\\ y_1 & y_2\end{bmatrix} = \begin{bmatrix}-1 & 4\\ 5 & -3\end{bmatrix}$$

(b) Find $[\vec{x}]_{\mathcal{C}}$ for $\vec{x} = 5\vec{b}_1 + 3\vec{b}_2$

The $\mathcal{B}$ coordinates of $\vec{x}$ are $\begin{bmatrix}\vec{b}_1 & \vec{b}_2\end{bmatrix}\begin{bmatrix}5\\3\end{bmatrix}$. Apply the change-of-coordinates matrix from (a)

$$[\vec{x}]_{\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}\leftarrow\mathcal{B}}\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}-1 & 4\\ 5 & -3\end{bmatrix}\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}5(-1)+3(4)\\ 3(5)+3(-3)\end{bmatrix}$$
$$[\vec{x}]_{\mathcal{C}} = \begin{bmatrix}-5+12\\ 15-9\end{bmatrix} = \begin{bmatrix}7\\6\end{bmatrix}$$

4.7.2 (3–4) Identify which of the given vector transformations is performed by the given change-of-coordinates matrix

3. Let $\mathcal{B} = \left\{\vec{b}_1, \vec{b}_2\right\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be bases for V, and let $\boldsymbol{P}$ be a matrix whose columns are $\left[\vec{b}_1\right]_{\mathcal{C}}$ and $\left[\vec{b}_2\right]_{\mathcal{C}}$. Which of the following equations is satisfied by $\boldsymbol{P}$ for all $\vec{x}$ in V?

$$[\vec{x}]_{\mathcal{B}} = \boldsymbol{P}\,[\vec{x}]_{\mathcal{C}}\,,\text{ or }\ [\vec{x}]_{\mathcal{C}} = \boldsymbol{P}\,[\vec{x}]_{\mathcal{B}}$$

We're asked to identify in which direction a change-of-coordinates matrix maps coordinates. If $\boldsymbol{P}$ is composed of the $\mathcal{B}$ vectors in $\mathcal{C}$ coordinates then does $\boldsymbol{P}$ change vectors from $\mathcal{B}$ coordinates to $\mathcal{C}$ coordinates or from $\mathcal{C}$ coordinates to $\mathcal{B}$ coordinates?

Consider that $\boldsymbol{P}$ is constructed by representing the basis vectors in $\mathcal{B}$ as their corresponding vectors in $\mathcal{C}$. We extend this notion to say that $\boldsymbol{P}$ represents *all* vectors in $\mathcal{B}$ as their corresponding vectors in $\mathcal{C}$. We write $\boldsymbol{P}$ then as $\boldsymbol{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ and the answer becomes clear:

$$[\vec{x}]_{\mathcal{C}} = \boldsymbol{P}\,[\vec{x}]_{\mathcal{B}}$$

4.7.3 (7–10) Find the change-of-coordinates matrices between two bases in both directions (find the matrix and its inverse)

Let $\mathcal{B} = \left\{\vec{b}_1, \vec{b}_2\right\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ be bases for $\mathbb{R}^2$. Find the change-of-coordinates-matrix from $\mathcal{B}$ to $\mathcal{C}$ and from $\mathcal{C}$ to $\mathcal{B}$.

7. $\vec{b}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}, \vec{c}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

Use the process of augmenting $\begin{bmatrix} \mathcal{C} & \mathcal{B} \end{bmatrix}$ and $\begin{bmatrix} \mathcal{B} & \mathcal{C} \end{bmatrix}$ and taking the reduced echelon forms. This configuration places the left-hand matrix of basis vectors in a position to be brought to the identity by row reduction; in doing so, we are establishing a context for the right-hand basis in which the left-hand basis is the "standard basis." For example in $\begin{bmatrix} \mathcal{C} & \mathcal{B} \end{bmatrix}$, rather than take vectors from $\mathcal{B}$ coordinates to E coordinates and from E to $\mathcal{C}$, we place the $\mathcal{B}$ vectors in a context wherein the $\mathcal{C}$ has become the "standard basis" and the usual $\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ matrix has transparently become a $\boldsymbol{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ matrix.

$$\begin{bmatrix} \mathcal{C} & \mathcal{B} \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ -5 & 2 & 5 & -1 \end{array}\right] \overset{\text{RREF}}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{array}\right]$$

$$\boldsymbol{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$$

We can find the matrix $\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{C}}$ by the same process, but it's analogous to taking the inverse $\boldsymbol{P}^{-1}_{\mathcal{C}\leftarrow\mathcal{B}}$. Both configurations are shown:

$$\begin{bmatrix}\mathcal{B} & \mathcal{C}\end{bmatrix} = \left[\begin{array}{cc|cc} 7 & -3 & 1 & -2 \\ 5 & -1 & -5 & 2 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 3 \end{array}\right]$$

$$\boldsymbol{P}^{-1}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}^{-1} = \left[\begin{array}{cc|cc} -3 & 1 & 1 & 0 \\ -5 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 3 \end{array}\right]$$

4.7.4 (11–12) Support or contradict statements about change-of-coordinates matrices

$\mathcal{B}$ and $\mathcal{C}$ are bases for a vector space V. Mark each statement *True* or *False* and justify your answer.

11. (a) The columns of the change-of-coordinates matrix $\boldsymbol{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ are $\mathcal{B}$-coordinate vectors of the vectors in $\mathcal{C}$.

False; the columns of the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ are $\mathcal{C}$-coordinate vectors of the vectors in $\mathcal{B}$. Consider by metaphor that the change-of-coordinates matrix is the transform of the axes in $\mathcal{B}$ to the axes in $\mathcal{C}$; when the axes are transformed the whole space measured by them is also transformed.

(b) If $V = \mathbb{R}^n$ and $\mathcal{C}$ is the *standard* basis for V, then $\boldsymbol{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ is the same as the change-of-coordinates matrix $\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}$.

True; by definition $\mathcal{E}$ is the standard basis for $\mathbb{R}^n$, so if V coincides with $\mathbb{R}^n$ and $\mathcal{C}$ is a standard basis for V then $\mathcal{C}$ is identical to $\mathcal{E}$.

4.7.5 (13–14) Find the change-of-coordinates matrix from the given basis in $\mathbb{P}_2$

13. In $\mathbb{P}_2$, find the change-of-coordinates matrix from the basis

$$\mathcal{B} = \left\{1 - 2t + t^2, 3 - 5t + 4t^2, 2t + 3t^2\right\}$$

to the standard basis $\mathcal{C} = \left\{1, t, t^2\right\}$. Then find the $\mathcal{B}$-coordinate vector for $-1 + 2t$.

The vectors in basis $\mathcal{B}$ are given in standard basis coordinates, so the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis $\mathcal{C}$ is

$$\boldsymbol{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

Converting the $\mathbb{P}_2$-vector $-1+2t$ from standard basis coordinates to $\mathcal{B}$ coordinates requires $\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{C}}$, the inverse of the matrix above:

$$\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{C}} = \boldsymbol{P}^{-1}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -23 & -9 & 6 \\ 8 & 3 & -2 \\ -3 & -1 & 1 \end{bmatrix}$$

The vector $-1+2t$ has standard basis coordinates $\vec{v} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$. Multiplying by the change-of-coordinates matrix we have

$$[\vec{v}]_{\mathcal{B}} = \boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{C}}\,(\vec{v}) = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

4.7.6 (19) Given a change-of-coordinates matrix $\boldsymbol{P}$ and the destination basis, find a basis for $\mathbb{R}^3$ that is transformed by $\boldsymbol{P}$ to that destination basis

19. Let $\boldsymbol{P} = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -5 & 0 \\ 4 & 6 & 1 \end{bmatrix}, \vec{b}_1 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} -8 \\ 5 \\ 2 \end{bmatrix}, \vec{b}_3 = \begin{bmatrix} -7 \\ 2 \\ 6 \end{bmatrix}$

(a) Find a basis $\mathcal{U} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ for $\mathbb{R}^3$ such that $\boldsymbol{P}$ is the change-of-coordinates matrix $\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{U}}$ from $\mathcal{U}$ to the basis $\mathcal{B} = \left\{\vec{b}_1, \vec{b}_2, \vec{b}_3\right\}$.

The normal way we'd construct $\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{U}}$, given the basis vectors in $\mathcal{U}$, would be to compose $\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{U}}$ and $\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{E}}$; in the shorthand form, we'd augment $\begin{bmatrix}\mathcal{B} & \mathcal{U}\end{bmatrix}$ and row reduce. We're given the reduced matrix and the destination basis $\mathcal{B}$, but the origin basis vectors $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ are unknown:

$$\left[\begin{array}{ccc|ccc} -2 & -8 & -7 & & & \\ 2 & 5 & 2 & \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ 3 & 2 & 6 & & & \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & -3 & -5 & 0 \\ 0 & 0 & 1 & 4 & 6 & 1 \end{array}\right]$$

Consider how this process produces $\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{U}}$: the destination basis is placed in the left half of the augmented matrix and reduced to the identity, as if that basis was the new standard basis. The right half of the augmented matrix then contains the origin basis $\mathcal{U}$ transformed relative to that new standard basis $\mathcal{B}$. The columns of $\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{U}}$ then are the origin basis vectors in $\mathcal{U}$ $(\vec{u}_1, \vec{u}_2, \vec{u}_3)$, but represented in $\mathcal{B}$ coordinates $([\vec{u}_1]_{\mathcal{B}}, [\vec{u}_2]_{\mathcal{B}}, [\vec{u}_3]_{\mathcal{B}})$. If we want those $\mathcal{U}$ vectors in true standard basis $\mathcal{E}$ coordinates then we can use the change-of-coordinates matrix $\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ to change $[\vec{u}_i]_{\mathcal{B}}$ to $\vec{u}_i$:

$$\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}} = \begin{bmatrix}\vec{b}_1 & \vec{b}_2 & \vec{b}_3\end{bmatrix} = \begin{bmatrix} -2 & -8 & -7 \\ 2 & 5 & 2 \\ 3 & 2 & 6 \end{bmatrix}$$

$$\begin{bmatrix}\vec{u}_1 & \vec{u}_2 & \vec{u}_3\end{bmatrix} = \boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{U}} = \begin{bmatrix} -6 & -6 & -5 \\ -5 & -9 & 0 \\ 21 & 32 & 3 \end{bmatrix}$$

$$\vec{u}_1 = \begin{bmatrix} -6 \\ -5 \\ 21 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -6 \\ -9 \\ 32 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}$$

(b) Find a basis $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ for $\mathbb{R}^3$ such that $\boldsymbol{P}$ is the change-of-coordinates matrix $\boldsymbol{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{C}$.

Again we have an augmented matrix of basis vectors, but this time the destination basis vectors are unknown:

$$\left[\begin{array}{ccc|ccc} & & & -2 & -8 & -7 \\ \vec{c}_1 & \vec{c}_2 & \vec{c}_3 & 2 & 5 & 2 \\ & & & 3 & 2 & 6 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 0 & -3 & -5 & 0 \\ 0 & 0 & 1 & 4 & 6 & 1 \end{array}\right]$$

The columns of $\boldsymbol{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ are $\left\{\left[\vec{b}_1\right]_{\mathcal{C}}, \left[\vec{b}_2\right]_{\mathcal{C}}, \left[\vec{b}_3\right]_{\mathcal{C}}\right\}$; if we take the inverse $\boldsymbol{P}_{\mathcal{C}\leftarrow\mathcal{B}}^{-1}$ then we have $\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{C}}$—and the columns of $\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{C}}$ are $\{[\vec{c}_1]_{\mathcal{B}}, [\vec{c}_2]_{\mathcal{B}}, [\vec{c}_3]_{\mathcal{B}}\}$:

$$\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{C}} = \boldsymbol{P}_{\mathcal{C}\leftarrow\mathcal{B}}^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ -3 & -5 & 0 \\ 4 & 6 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & 8 & 5 \\ -3 & -5 & -3 \\ -2 & -2 & -1 \end{bmatrix}$$

Transforming $\{[\vec{c}_1]_{\mathcal{B}}, [\vec{c}_2]_{\mathcal{B}}, [\vec{c}_3]_{\mathcal{B}}\}$ to standard basis $\mathcal{E}$ coordinates then requires $\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}$ from part (a):

$$\boldsymbol{P}_{\mathcal{E}\leftarrow\mathcal{B}}\boldsymbol{P}_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 28 & 38 & 21 \\ -9 & -13 & -7 \\ -3 & 2 & 3 \end{bmatrix}$$

$$\vec{c}_1 = \begin{bmatrix} 28 \\ -9 \\ -3 \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} 38 \\ -13 \\ 2 \end{bmatrix}, \vec{c}_3 = \begin{bmatrix} 21 \\ -7 \\ 3 \end{bmatrix}$$

4.8 Extra Topic: Error Correcting Codes

Chapter 5

Eigenvalues and Eigenvectors

5.1 Eigenvectors and Eigenvalues

Exercises to practice

1–26, 37–40

☐☐☐ **5.1.1** **(1–2)** ***(Definition)*** **Determine if the given value is an eigenvalue of the given matrix**

☐☐☐ **5.1.2** **(3–6)** ***(Definition)*** **Determine if the given vector is an eigenvector of the matrix and find the eigenvalue**

☐☐☐ **5.1.3** **(7–8)** ***(Definition)*** **Determine if the given value is an eigenvalue of the matrix and find one eigenvector; how many eigenvectors are there?**

☐☐☐ **5.1.4** **(9–16)** ***(Practice)*** **Find a basis for the eigenspace given an eigenvalue of the matrix**

☐☐☐ **5.1.5** **(17–19)** ***(Practice)*** **Find the eigenvalues of the given matrices; how many eigenvalues are there?**

☐☐☐ **5.1.6** **(20)** ***(Concept)*** **Find one eigenvalue and two eigenvectors for the matrix without calculations; what concept is sufficient to find the answer?**

☐☐☐ **5.1.7** **(21–22)** ***(Concept)*** **Support or contradict statements about eigenvalues and eigenvectors**

Questions to ask

☐☐☐ **5.1.8** **(23)** ***(Concept)*** **Explain the correspondence between the dimension of a matrix and the number of its unique eigenvalues**

☐☐☐ **5.1.9** **(24)** ***(Concept)*** **Give an example of a** 2×2 **matrix with one distinct eigenvalue; see Exercise 23 to justify your answer**

☐☐☐ **5.1.10** **(25)** ***(Concept)*** **Prove the relationship between the inverse of a matrix and the inverse of its eigenvalue**

☐☐☐ **5.1.11** **(26)** ***(Concept)*** **Define a nilpotent matrix; prove that the eigenvalue of a nilpotent matrix of index** 2 **is** 0

☐☐☐ **5.1.12** **(37–40)** ***(Practice)*** **Use a calculator to find the eigenvalues of the given matrix; then use row reduction to find a basis for each eigenspace**

5.2 The Characteristic Equation

Exercises to practice

1–22

☐☐☐ **5.2.1** **(1–8)** ***(Practice)*** **Find the characteristic polynomial and eigenvalues of the matrix**

☐☐☐ **5.2.2** **(9–14)** ***(Practice)*** **Find the characteristic polynomial of the 3×3 matrix using *cofactor expansion* from Section 3.1 (see homework on page ??, Lay p. 168)**

☐☐☐ **5.2.3** **(15–17)** ***(Concept)*** **List the eigenvalues of the matrix, repeated according to their multiplicity (number of occurrences in the factors of the characteristic polynomial)**

☐☐☐ **5.2.4** **(18)** ***(Concept)*** **Understand the relationship between the multiplicity of an eigenvalue and the dimension of an eigenspace; choose a value for an entry in the matrix such that the 5-eigenspace has a specific dimension**

☐☐☐ **5.2.5** **(19)** ***(Concept)*** **Illustrate how the eigenvalues of a matrix are multiplicative factors of its determinant**

Questions to ask

☐☐☐ **5.2.6** **(20)** ***(Concept)*** **Choose the appropriate property of determinants to show that a matrix $\boldsymbol{A}$ and its transpose $\boldsymbol{A}^T$ have the same characteristic polynomial**

☐☐☐ **5.2.7** **(21–22)** ***(Concept)*** **Support or contradict statements about the eigenvalues and characteristic polynomial of a matrix**

5.3 Diagonalization

Exercises to practice

1–27, 33–36

☐☐☐ **5.3.1** **(1–4)** ***(Practice)*** **Use the diagonalization theorem (5 in Lay p. 284) to efficiently compute the** 4**th or** k**th power of the given matrix**

☐☐☐ **5.3.2** **(5–6)** ***(Practice)*** **Use the diagonalization theorem to find the eigenvalues of the matrix and a basis for each eigenspace**

☐☐☐ **5.3.3** **(7–20)** ***(Practice)*** **Diagonalize the matrix if possible, or state why it can't be diagonalized**

☐☐☐ **5.3.4** **(21–22)** ***(Concept)*** **Support or contradict statements about diagonalizable matrices (see Theorems 5 and 6 in Lay pp. 284 and 286)**

☐☐☐ **5.3.5** **(23–26)** ***(Concept)*** **Determine if the matrix is diagonalizable from the dimensions of its eigenspaces**

☐☐☐ **5.3.6** **(27)** ***(Concept)*** **Prove that if** $\boldsymbol{A}$ **is diagonalizable and invertible then so is** $\boldsymbol{A}^{-1}$

Questions to ask

☐☐☐ **5.3.7** **(33–36)** ***(Practice)*** **Use a calculator to diagonalize the given matrix; then find the eigenvalues and bases for the eigenspaces**

5.5 Complex Eigenvalues

Exercises to practice

1–6

☐☐☐ **5.5.1 (1–6)** ***(Practice)*** **Find the eigenvalues and a basis for each eigenspace in** $\mathbb{C}^2$

Questions to ask

Chapter 6

Orthogonality and Least Squares

6.1 Inner Product, Length, and Orthogonality

Exercises to practice

1–20, 23, 24, 26, 29

☐☐☐ **6.1.1** **(1–8) Compute the inner products and quotients of products of the given vectors**

☐☐☐ **6.1.2** **(9–12) Normalize the given vector (find a unit vector in the same direction)**

☐☐☐ **6.1.3** **(13–14) Find the distance between the given vectors**

☐☐☐ **6.1.4** **(15–18) Determine which pairs of vectors are orthogonal (their dot product is zero)**

☐☐☐ **6.1.5** **(19–20) Contradict or support statements about the dot product of vectors in $\mathbb{R}^n$**

☐☐☐ **6.1.6** **(23) Compute the length of a sum of vectors and compare its equation to the Pythagorean Theorem**

☐☐☐ **6.1.7** **(24) Apply the *parallelogram law* to add vectors in $\mathbb{R}^n$**

☐☐☐ **6.1.8** **(26) Apply a theorem from Chapter 4 (Vector Spaces) to show that the space of vectors orthogonal to a given vector is a subspace of $\mathbb{R}^3$**

Questions to ask

☐☐☐ **6.1.9** **(29) Prove that if a vector is orthogonal to every basis vector for a vector space then it is orthogonal to every vector in that space**

6.2 Orthogonal Sets

Exercises to practice

1–24, 35

☐☐☐ **6.2.1 (1–6) Determine orthogonal sets of vectors**

☐☐☐ **6.2.2 (7–10) Make an orthogonal basis for $\mathbb{R}^2$ or $\mathbb{R}^3$ from the given vectors and express $\vec{x}$ in terms of that basis**

☐☐☐ **6.2.3 (11–12) Compute orthogonal projections of one vector onto another**

☐☐☐ **6.2.4 (13–14) Compute orthogonal projections of one vector onto a plane**

☐☐☐ **6.2.5 (15–16) Compute the shortest distance between a vector and a line (the length of the ray orthogonal to the line)**

☐☐☐ **6.2.6 (17–22) Identify orthonormal vectors, or normalize the given orthogonal vectors**

☐☐☐ **6.2.7 (23–24) Support or contradict statements about orthogonal vectors**

☐☐☐ **6.2.8 (35) Use a calculator to identify orthogonal vectors in the columns of the given matrix (*hint:* consider the product A^TA)**

Questions to ask

6.3 Orthogonal Projections

Exercises to practice

1–12

☐☐☐ **6.3.1** **(1–2) Construct an orthogonal basis for a 2-dimensional subspace of $\mathbb{R}^4$ and express the vector $\vec{x}$ in terms of that basis**

☐☐☐ **6.3.2** **(3–6) Find the orthogonal projection of $\vec{y}$ onto the plane formed by the given vectors, if they are orthogonal**

☐☐☐ **6.3.3** **(7–10) Decompose $\vec{y}$ into its parallel and orthogonal components with regard to a subspace of $\mathbb{R}^3$**

☐☐☐ **6.3.4** **(11–12) Find the point in a subspace of $\mathbb{R}^4$ that is closest to a vector not in that subspace (*hint:* as in Exercises 15–16 in 6.2.5 on page 69, consider the orthogonal distance to the subspace)**

Questions to ask

6.4 The Gram-Schmidt Process

Exercises to practice

1–12, 24

☐☐☐ **6.4.1** **(1–6) Use the Gram-Schmidt process to convert the given basis to an orthogonal one**

☐☐☐ **6.4.2** **(7–8) Normalize the orthogonal bases found in Exercises 3–4 to make orthonormal bases**

☐☐☐ **6.4.3** **(9–12) Find an orthogonal basis for the column space of the given matrix**

☐☐☐ **6.4.4** **(24) Using a calculator, apply the Gram-Schmidt process to find an orthogonal basis for the column space of the given matrix**

Questions to ask