MAT 231 — Calculus II

Homework and Skills Overview

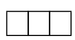
Avilez, Fall 2019

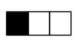
Calculus: Early Transcendentals, 7th Edition, James Stewart

How to use this outline

The homework is divided into chapters and sections according to the book. Each section is further divided into groups of exercises related by their focus on a particular skill or method.

Beside each group of exercises is a bar of three blocks that indicate where attention should be focused. In order from most to least important, these indications are

 never attempted;

 not adequately prepared;

 need practice;

 generally comfortable, but specific questions need practice; and

 comfortable.

Homework pages are numbered according to the chapter (Ch), section (§), subsection (§§), and page of work (p) in that subsection, written as

$$Ch.\S.\S\S - p\,.$$

Contents

Chapter 6

Applications of Integration

6.1 Areas Between Curves

1–32

Exercises to practice

☐☐☐ **6.1.1** **(1–4) Integrate the area between two curves whose graphs are shown; choose the variable of integration to give the easiest solution. The interval of integration is given.**

☐☐☐ **6.1.2** **(5–12) Sketch the curves whose functions are given and choose a variable of integration. Illustrate an example of an approximating rectangle and label its dimensions. Integrate the area between the curves using the infinitesimal case of that rectangle. The interval is not given and must be found by solving intersections of the curves.**

☐☐☐ **6.1.3** **(13–28) Sketch the curves whose functions are given and solve their intersections. Integrate the area enclosed by the curves.**

☐☐☐ **6.1.4** **(29–30) Integrate the area of a triangle with the given vertices.**

☐☐☐ **6.1.5** **(31–32) Evaluate the given integral and interpret its value as the area of region. Sketch the region (work backward by reading an integral to identify the curves being integrated).**

Questions to ask

31. This exercise involves *triginometric identities* to solve the intersections. It also has us integrating the absolute value of a function; after solving the intersections we use them to separate the function into different intervals of integration—areas with negative values are made positive as the absolute value would do. We're expected to know common values of sin and cos and to be able to do algebra with them using identities, including factoring polynomials.

6.2 Volumes

Exercises to practice

1–34, 39–42

☐☐☐ **6.2.1** **(1–18) Sketch the given curves and the intersected solid; identify the primitive of integration and its parameters; identify the interval and integrate primitives to find the volume of the solid.**

☐☐☐ **6.2.2** **(19–30) Revolve the given region about various axes and integrate the volume of the swept solid.**

☐☐☐ **6.2.3** **(31–34) (Approximation) Set up an integral for the volume of a solid as given before, but approximate its value with a calculator, accurate to five decimal places.**

☐☐☐ **6.2.4** **(39–42) (Working backward) Describe the solid whose integral of volume is given (use previous skills to deduce the conditions that produce the given integral).**

Questions to ask

6.3 Volumes by Cylindrical Shells

Exercises to practice

1–26, 29–32, 37–43

☐☐☐ **6.3.1** **(1–2) Explain why the methods from Sections 6.1 and 6.2 are inconvenient for finding the volumes of the given solids (the curve of revolution is given).**

☐☐☐ **6.3.2** **(3–7) Use cylindrical shells as the primitive of integration to find the volume enclosed by the given revolute surfaces (revolved about the y-axis).**

☐☐☐ **6.3.3** **(8) Use both slicing and cylindrical shells to find the volume enclosed by the given revolute surface. Sketch each method and understand the differences between them.**

☐☐☐ **6.3.4** **(9–14) Use cylindrical shells to integrate the volume enclosed by the given revolute surfaces (revolved about the x-axis).**

☐☐☐ **6.3.5** **(15–20) Use cylindrical shells to integrate the volume enclosed by the given curves when revolved about an axis *offset* from the x- or y-axis.**

Questions to ask

☐☐☐ **6.3.6** **(21–26) (Approximation) Set up the volume integral as before, but use a calculator to approximate its value accurate to five decimal places.**

☐☐☐ **6.3.7** **(29–32) (Working backward) Describe the solid whose integral of volume is given (use previous skills to deduce the conditions that produce the given integral).**

☐☐☐ **6.3.8** **(37–43) (Big picture) Use any method from Sections 6.2 and 6.3 to integrate the volume of the solid enclosed by the given revolute surfaces. Axes of revolution include the x- and y-axes and offsets of them.**

6.4 Work

Exercises to practice

2–5, 7–12

☐☐☐ **6.4.1** **(2) Find the work done when the force is constant**

☐☐☐ **6.4.2** **(3–4) Find the work done when the force is given as a function of distance**

☐☐☐ **6.4.3** **(5) Find the work done by computing the area under a graph of linear functions**

☐☐☐ **6.4.4** **(7–8) Find the rate of a spring from the description of its reaction to a given force; use that spring rate to calculate the work done stretching the spring to another length**

☐☐☐ **6.4.5** **(9–10) Use the given work done to a spring to find its rate; use that rate to predict its reaction to (a) a stretch of given length and (b) a given force**

☐☐☐ **6.4.6** **(11) Compare the work needed to elicit different reactions from a spring; understand the relationship between work and spring rate**

☐☐☐ **6.4.7** **(12) Use the work needed to produce the given stretch of a spring to find its rate**

Questions to ask

6.5 Average Value of a Function

Exercises to practice

1–10, 13, 14

☐☐☐ **6.5.1 (1–8) (Practice) Find the average value of nonlinear functions**

☐☐☐ **6.5.2 (9–10, 13) (Proof of concept) Show that the average value of a function occurs in the range of that function**

☐☐☐ **6.5.3 (14) (Working backward) Find the interval on which a function has the given average value**

Questions to ask

Chapter 7

Techniques of Integration

7.1 Integration by Parts

Exercises to practice

1–42

The LIATE principle for choosing u from factors in an integrand

Choose u from the first available factor in this list:

Logarithmic

Inverse Trigonometric

Algebraic

Trigonometric

Exponential

For example, in $\int xe^{2x}dx$ the integrand is the product of an algebraic function (x) and an exponential function (e^{2x}). Referring to the list, we find Algebraic before Exponential and so choose $u = x$ and $dv = e^{2x}dx$.

☐☐☐ **7.1.1 (1, 2) Integrate by parts given the chosen u and dv**

☐☐☐ **7.1.2 (3–36) Evaluate the integral with integration by parts**

☐☐☐ **7.1.3 (37–42) Transform the integral so that it can be integrated by parts**

Questions to ask

7.2 Trigonometric Integrals

Exercises to practice

1–40, 44–49

☐☐☐ **7.2.1 (1–40, 44–49) Evaluate the integral of trigonometric functions**

Questions to ask

7.3 Trigonometric Substitution

Exercises to practice

1–30

☐☐☐ **7.3.1 (1–3) Evaluate the integral by making the given trig substitution**

☐☐☐ **7.3.2 (4–30) Find an appropriate trig substitution and integrate**

Questions to ask

7.4 Integration of Rational Functions by Partial Fractions

Exercises to practice

1–51

☐☐☐ **7.4.1 (1–6) Apply partial fraction decomposition to the function**

☐☐☐ **7.4.2 (7–38) Use partial fraction decomposition to evaluate the integral**

☐☐☐ **7.4.3 (39–51) Transform the integrand into a rational function and integrate with partial fraction decomposition**

Questions to ask

7.5 Strategy for Integration

Exercises to practice

1–82 (except 35, 53)

☐☐☐ **7.5.1 (1–82, ~~35~~, ~~53~~) Practice all basic integration techniques**

☐☐☐ **7.5.2 Substitution, Integration by Parts (2, 5, 7, 9, 18–20, 23, 27, 29–31, 33, 40, 34–48, 52, 55–57, 63, 64, 67, 71, 72, 74, 76, 77)**

☐☐☐ **7.5.3 Trig Integrals (1, 3, 4, 8, 13, 21, 28, 34–39, 41–42, 59, 61, 62, 65–66, 73, 78–82)**

☐☐☐ **7.5.4 Trig Substitution (14–16, 22, 40, 49–51, 58, 60, 69, 73, 77)**

☐☐☐ **7.5.5 Partial Fraction Decomposition (6, 10–12, 24–26, 32, 43, 52, 67, 68, 75)**

Questions to ask

7.7 Approximate Integration

Exercises to practice

1, 2, 5–22

☐☐☐ **7.7.1 (1) Graphically understand the approximations of an integral**

☐☐☐ **7.7.2 (2) Show understanding of approximations by identifying them**

☐☐☐ **7.7.3 (5, 6) Approximate with the Midpoint and Simpson's rules**

☐☐☐ **7.7.4 (7–18) Use the Trapezoidal, Midpoint, and Simpson's rules**

☐☐☐ **7.7.5 (19–22) Estimate error and attain a specified precision**

20. (a) Find the approximations T_{10} and M_{10} for $\int_1^2 e^{1/x} dx$.

The trapezoidal approximation T_n is defined by

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2\left(f(x_1) + \cdots + f(x_{n-1})\right) + f(x_n)\right]$$

Questions to ask

where $\Delta x = \frac{b-a}{n}$, b and a being the upper and lower bounds of integration respectively, and n being the number of subintervals over which to sample the function. In our case $f(x) = e^{1/x}$, $a = 1$, $b = 2$, and $n = 10$, so $\Delta x = \frac{2-1}{10} = \frac{1}{10}$. The values of these variables will be the same for both approximation methods.

The values of x for each subinterval i are given by $a + i\Delta x$ where $0 \leq i \leq n$, i.e. $\{1, 1 + \frac{1}{10}, 1 + \frac{2}{10}, \cdots, 2\}$. Substituting in the definition of T_n gives us

$$T_{10} = \frac{1}{2 \times 10}\left[e^{10/11} + 2\left(e^{5/6} + e^{10/13} + \cdots + e^{10/19}\right) + e^{1/2}\right]$$

$$T_{10} \approx 2.02197\ldots$$

This value of T_{10} is numerically approximated by a Python language program which is included in Listing 1 on page 23.

The midpoint approximation M_n is different in a couple of ways, most notably in how the values of x are defined for each subinterval:

$$\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i),\ 1 \leq i \leq n$$

where again $x_i = a + i\Delta x$. This means that the function is sampled at the midpoint of each subinterval, giving the midpoint approximation its name.

The midpoint approximation is defined by

$$M_n = \Delta x \left[f(\overline{x}_1) + f(\overline{x}_1) + \cdots + f(\overline{x}_n) \right]$$

and substituting we have

$$M_{10} = \frac{1}{10}\left[e^{20/21} + e^{20/23} + \cdots + e^{20/39} \right]$$

$$M_{10} \approx 2.01910\ldots$$

The program approximating this value of M_{10} can be seen in Listing 2 on the following page.

(b) Estimate the errors in T_{10} and M_{10} in (a) The magnitude of the error in the trapezoidal and midpoint approximations is given respectively by the inequalities

$$|E_T| \leq \frac{k(b-a)^3}{12n^2}$$

$$|E_M| \leq \frac{k(b-a)^3}{24n^2}$$

where k is an integer such that $|f''(x)| \leq k$. Taking the second derivative of $f(x) = e^{1/x}$ we have

$$f'(x) = -\frac{1}{x^2} e^{1/x}$$

$$f''(x) = e^{1/x}\left(\frac{1}{x^4} + \frac{2}{x^3} \right)$$

$$\left| e^{1/x}\left(\frac{1}{x^4} + \frac{2}{x^3} \right) \right| \leq k$$

we could find the relative maximum of $|f''(x)|$ by solving $\left|f^3(c)\right| = 0$ and taking the largest of $\left|f^3(x)\right|, x \in \{a, c, b\}$; but for the sake of time we determine the maximum graphically

$$\max \left[|f''(x)| \right]_a^b = |f''(a)| = 8.155 \leq k \in \mathbb{Z}$$

$$k \geq 9$$

Substituting in the error estimate inequalities we have

$$|E_T| \leq \frac{9(2-1)^3}{12 \times 10^2} = 0.00750$$

$$|E_M| \leq \frac{9(2-1)^3}{24 \times 10^2} = 0.00375$$

so we see that both approximations are accurate to within 7500 parts per million, with the midpoint approximation being at worst twice as accurate as the trapezoidal approximation.

Listing 1: Numerically approximating the value of T_{10} for $\int_1^2 e^{1/x} dx$

```
# Python 3.6.0
from math import exp
a, b, n = 1, 2, 10
dx = (b-a)/n
Tcoeff = lambda i: 2 if i > 0 and i < n else 1
xi = map(lambda i: (a + i*dx, Tcoeff(i)), range(n+1))
f = lambda xc: exp(1 / xc[0]) * xc[1]
S = dx/2 * sum(map(f, xi))
print(S)

2.0219756440513623
```

Listing 2: Numerically approximating the value of M_{10} for $\int_1^2 e^{1/x} dx$

```
# Python 3.6.0
from math import exp
a, b, n = 1, 2, 10
dx = (b-a)/n
midpoint = lambda i: (a+(i-1)*dx + a+i*dx)/2
xi = map(midpoint, range(1, n+1))
f = lambda x: exp(1 / x)
S = dx * sum(map(f, xi))
print(S)

2.019101884886549
```

(c) How large do we have to choose n so T_n and M_n are accurate to within 0.0001? We find the necessary value of n by setting the desired accuracy in the inequalities above and solving for n:

$$|E_T| \leq \frac{k(b-a)^3}{12n_T^2} \leq 0.0001$$

$$\begin{aligned}\frac{9}{12 \times 0.0001} &\leq n_T^2 \\ \sqrt{7500} = 86.6\ldots &\leq n_T \\ n_T &= 87\end{aligned}$$

and

$$|M_T| \leq \frac{k(b-a)^3}{24n_M^2} \leq 0.0001$$

$$\begin{aligned}\sqrt{\frac{9}{24 \times 0.0001}} &= 61.2\ldots \leq n_M \\ n_M &= 62\end{aligned}$$

Inputting these values of n_T and n_M into the programs in Listings 1 and 2 respectively we have

$$\begin{aligned}T_{10} &\approx 2.02003\ldots \\ M_{10} &\approx 2.02008\ldots\end{aligned}$$

which are consistent to within 0.0001 or 100 parts per million.

7.8 Improper Integrals

Exercises to practice

1–3, 5–40, 49–54

☐☐☐ **7.8.1 (1) Explain what makes the given integrals improper**

☐☐☐ **7.8.2 (2) Identify proper and improper integrals**

☐☐☐ **7.8.3 (3) Use an improper integral to calculate area as** $x \to \infty$

☐☐☐ **7.8.4 (5–40) Determine whether the improper integral converges or diverges and evaluate it if it converges**

☐☐☐ **7.8.5 (49–54) Use the *Comparison Theorem* to determine whether the improper integral converges or diverges**

Questions to ask

Chapter 8

Applications of Integration

8.1 Arc Length

Exercises to practice

1–20, 33–35

☐☐☐ **8.1.1** **(1, 2) Compare the *Arc Length Formula* with understood functions**

☐☐☐ **8.1.2** **(3–6) Numerically approximate the arc length formula**

☐☐☐ **8.1.3** **(7–18) Integrate to find the exact length of a curve**

☐☐☐ **8.1.4** **(19, 20) Integrate the length of a curve between two points**

☐☐☐ **8.1.5** **(33–35) Construct a function of arc length for a curve from a given starting point**

Questions to ask

8.2 Area of a Surface of Revolution

Exercises to practice

1–16, 25, 26

☐☐☐ **8.2.1 (1–4) Set up and numerically integrate the area of a revolute surface**

☐☐☐ **8.2.2 (5–12) Evaluate the area of a revolute surface about the x-axis**

☐☐☐ **8.2.3 (13–16) Evaluate the area of a revolute surface about the y-axis**

☐☐☐ **8.2.4 (25) Revolve $1/x$ and understand its infinite surface area and finite volume. Consider the divergent harmonic series and its convergent terms**

☐☐☐ **8.2.5 (26) Revolve an infinite curve about the x-axis and observe its area**

Questions to ask

8.3 Applications in Physics and Engineering

Exercises to practice

21–35

☐☐☐ **8.3.1** **(21, 22) Find the moment M and the center of mass $\overline{x}$ on a line**

☐☐☐ **8.3.2** **(23, 24) Find the moment and center of mass on a plane**

☐☐☐ **8.3.3** **(25–33) Find the location of the centroid given bounding curves**

☐☐☐ **8.3.4** **(34, 35) Find the moment and center of mass given bounding curves**

Questions to ask

Chapter 10

Parametric Equations and Polar Coordinates

10.1 Curves Defined by Parametric Equations

Exercises to practice

1–22, 28, 37, 38

☐☐☐ **10.1.1** **(1–4) Practice sketching parametric curves by plotting points; reinforce directionality while sketching**

☐☐☐ **10.1.2** **(5–10) Practice sketching parametric curves by plotting; convert to Cartesian form by parameter elimination**

☐☐☐ **10.1.3** **(11–18) Convert to Cartesian form by parameter elimination, then sketch curves by plotting points**

☐☐☐ **10.1.4** **(19–22) Describe the trajectory along a parametric curve as the parameter varies**

☐☐☐ **10.1.5** **(28) Without graphing, match each parametric equation to the image of its curve**

☐☐☐ **10.1.6** **(37–38) Describe the differences between the curves of the given parametric equations**

Questions to ask

10.2 Calculus with Parametric Curves

Exercises to practice

1–20, 23–25, 28–34, 37–48, 57–63, 65, 66

☐☐☐ **10.2.1** **(1–2) Practice differentiating simple parametric equations**

☐☐☐ **10.2.2** **(3–6) Find an equation for the tangent line at a point on the parametric curve**

☐☐☐ **10.2.3** **(7–8) Practice differentiating normally, by parameter elimination, and differentiating x and y separately**

☐☐☐ **10.2.4** **(9–10) Find an equation for the tangent line and graph the curve and that line**

☐☐☐ **10.2.5** **(11–16) Take the first and second derivatives of the parametric equation and determine the concavity of its curve**

☐☐☐ **10.2.6** **(17–20) Find the critical points of the given parametric equation**

☐☐☐ **10.2.7** **(23–24) Graph the given parametric equation and determine an appropriate viewing window. Why is choosing the right window for this equation important?**

Questions to ask

☐☐☐ **10.2.8** **(25) Find the equations of two tangent lines for the given equation; sketch the curve and tangent lines**

☐☐☐ **10.2.9** **(28–30) Find the slope of the tangent to the given curve and identify critical points of that tangent**

☐☐☐ **10.2.10** **(31–34) Find the area enclosed by the parametric equation**

☐☐☐ **10.2.11** **(37–40) Set up the integral for the length of the parametric curve and approximate its value numerically**

☐☐☐ **10.2.12** **(41–44) Find the exact length of the parametric curve**

☐☐☐ **10.2.13** **(45–46) Graph the parametric curve and find its exact length**

☐☐☐ **10.2.14** **(47) Graph the curve and approximate its length numerically**

☐☐☐ **10.2.15** **(48) Find the length of the portion of the curve that forms a loop**

☐☐☐ **10.2.16** **(57–60) Set up the integral for the are of the revolute surface and approximate its value numerically**

☐☐☐ **10.2.17** **(61–63) Find the exact area of the surface of revolution about the x-axis**

☐☐☐ **10.2.18** **(65–66) Find the exact area of the surface of revolution about the y-axis**

10.3 Polar Coordinates

Exercises to practice

1–46, 54–72

☐☐☐ **10.3.1 (1–2) Plot the point by the given polar coordinates and give two equivalent points**

☐☐☐ **10.3.2 (3–4) Plot the point by the given polar coordinates and give the Cartesian coordinates for it**

☐☐☐ **10.3.3 (5–6) Give two equivalent sets of polar coordinates for point whose Cartesian coordinates are given**

☐☐☐ **10.3.4 (7–12) Sketch the region that contains the points with the given constraints on their polar coordinates**

☐☐☐ **10.3.5 (13–14) Find the distance between two points whose polar coordinates are given; find a general formula for that distance**

☐☐☐ **10.3.6 (15–20) Find a Cartesian equation for the curve described by the given polar equation**

☐☐☐ **10.3.7 (21–26) Find a polar equation for the curve described by the given Cartesian equation**

Questions to ask

☐☐☐ **10.3.8 (27–28) Decide whether the given curve is better described by a polar or Cartesian equation and give that equation**

☐☐☐ **10.3.9 (29–46) Sketch the curve described by the given polar equation by first sketching a Cartesian graph of $r(\theta)$**

☐☐☐ **10.3.10 (54) Match the polar equation to its graph**

☐☐☐ **10.3.11 (55–60) Find the slope of the tangent line to the given polar curve at the point specified by θ**

☐☐☐ **10.3.12 (61–64) Find the critical points on the given polar curve**

□□□ **10.3.13** **(65) Prove that the given polar equation is the equation of a circle and find its center and radius**

□□□ **10.3.14** **(66) Prove that the given polar curves intersect at right angles (*hint:* compare their tangent lines at the point of intersection)**

□□□ **10.3.15** **(67–72) Graph the given polar curve and choose an appropriate window to view it. Why is the chosen window important for viewing this curve?**

10.4 Areas and Lengths in Polar Coordinates

Exercises to practice

1–42, 45–48

☐☐☐ **10.4.1 (1–4) Find the area of a region bounded by a curve described by a polar equation**

☐☐☐ **10.4.2 (5–8) Find the area of a region by first finding the limits of integration from a graph of the equation**

☐☐☐ **10.4.3 (9–12) Sketch from memory the curve described by the given polar equation; then find the area it encloses**

☐☐☐ **10.4.4 (13–16) Graph the curve and find the area it encloses**

☐☐☐ **10.4.5 (17–22) Find the area enclosed by one loop of the curve by finding the limits of integration as intersections of the curve**

☐☐☐ **10.4.6 (23-28) Find the area of the region between two curves**

☐☐☐ **10.4.7 (29–34) Find the area of the region intersected by two curves**

Questions to ask

☐☐☐ **10.4.8 (35–36) Find the area between two loops of the same curve**

☐☐☐ **10.4.9 (37–42) Find all intersections of the given curves**

☐☐☐ **10.4.10 (45–48) Find the exact length of the polar curve**

Chapter 11

Infinite Sequences and Series

11.1 Sequences

Exercises to practice

1–18, 23–63, 70b, 72–79, 80b, 81, 82

☐☐☐ **11.1.1 (1, 2) Define sequences**

☐☐☐ **11.1.2 (3–12) List entries of a sequence given its definition**

☐☐☐ **11.1.3 (13–18) Find a formula for a_n that generates the given entries**

☐☐☐ **11.1.4 (23–56) Determine sequence convergence; find limits of sequences**

☐☐☐ **11.1.5 (57–63) Analyze sequences graphically; guess and find their limits**

☐☐☐ **11.1.6 (70b) A recursive sequence: $a_{n+1} = 1/(1+a_n)$**

70. (b) A sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$ for $n \geq 1$. Assuming that $\{a_n\}$ is convergent, find its limit.

This problem reminds us that a limit is an algebraic object that can be named and manipulated. Notice that the limit of a_n and a_{n+1} must be the same, and capture that limit in L:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{1+a_n} = L$$

Questions to ask

Using the relationship $\lim_{n\to\infty} a_n = L$, and remembering properties of limits,

$$\lim_{n\to\infty} \frac{1}{1+a_n} = \frac{1}{1+\lim_{n\to\infty} a_n} = \frac{1}{1+L} = L$$

this reminds us that the limit is not a specific value that the sequence will actually achieve, but it is also not a dead symbol; it can be named and participate in algebra by properties of limits. Rearranging, we have

$$\begin{aligned} 1 &= L(1+L) \\ 0 &= L^2 + L - 1 \end{aligned}$$

which is not easy to factor, so we use the quadratic formula. Notice that a prominent feature of Calc II is requiring us to remember and call upon many different areas of our math education without being explicitly told which we will need.

$$L = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$L = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)}$$
$$L = \frac{-1 \pm \sqrt{5}}{2}$$

Now, because L is the limit of a positive sequence it cannot then have a negative value. So we ignore the negative result for L (*what does it mean to say that this limit has a negative solution?)*

$$L = \lim_{n\to\infty} a_n = \frac{-1 + \sqrt{5}}{2}$$

□□□ 11.1.7 (72–78) Monotonic and bounded sequences

The exercises in this group are straightforward: is the given sequence monotonic? That is, are its entries always increasing, always decreasing, or staying the same? If it is monotonic, is it bounded? This is an important observation to be able to make because the *monotonic convergence theorem* tells us that every bounded monotonic sequence converges. Understanding that behavior is a necessary insight into what it means when a sequence has or doesn't have a finite limit.

72. $a_n = (-2)^{n+1}$

This is our first example of a non-monotonic sequence. We can see that it does not always increase or always decrease by writing out terms:

$$a_1 = (-2)^2 = 4$$
$$a_2 = (-2)^3 = -8$$
$$a_3 = (-2)^4 = 16$$

Not only are the entries increasing apparently without bound, but they alternate sign while growing, so they are becoming both increasingly positive as well as increasingly negative. Clearly this sequence is blowing up to $\pm\infty$.

---

74. $a_n = \frac{2n-3}{3n+4}$

Determine the direction of this sequence by considering the derivative of the function $f(x)$ where $f(n) = a_n$ when n is a positive integer (*function theorem* *of sequences).*

$$\begin{aligned} f(x) &= \frac{2x-3}{3x+4} \\ f'(x) &= \frac{2(3x+4)-3(2x-3)}{(3x+4)^2} \\ &= \frac{6x+8-6x+9}{(3x+4)^2} \\ &= \frac{17}{(3x+4)^2} > 0 \end{aligned}$$

The derivative is strictly positive so the function is increasing (though at a decreasing rate), and when the function is evaluated at integers n it gives us the strictly increasing entries of a_n.

To show the bounds of the sequence we use a skill that will be useful in evaluating series: determining comparable sequences. To show a lower bound we only need to show that the sequence entries a_n are always larger than a_1 when $n > 1$, which we proved by showing that the sequence increases. So we say the lower bound is $a_1 = -\frac{1}{7}$.

The upper bound can be found using the *comparison theorem of sequences,* which requires us to choose good sequences to compare a_n with (something we will do often to determine series convergence).

$$a_n = \frac{2n-3}{3n+4} < \frac{2n-3}{3n} < \frac{2n}{3n} < \frac{2}{3}$$

Additionally, taking the limit is trivial using *L'Hopital's Rule:*

$$\lim_{n\to\infty} \frac{2n-3}{3n+4} \overset{\text{H}}{=} \lim_{n\to\infty} \frac{2}{3} = \frac{2}{3}$$

---

75. $a_n = n(-1)^n$

Another sequnce that changes sign, this one grows by the always-positive coefficient n, so while it changes sign it also grows in absolute magnitude. We can see this sequence alternating to $\pm\infty$ and so it is not monotonic and has no finite limit (or bound). *Remember:* the limit of an oscillating function does not exist. Let $f(x) = (-1)^x$; does this function have a limit?

---

76. $a_n = ne^{-n} = \frac{n}{e^n}$

Given the speeds of functions we can see that the exponential denominator quickly dominates the linear numerator, so this function decreases monotonically (and quickly). Prove this by again taking the derivative of $f(x)$ using the function theorem of sequences

$$\begin{aligned} f(x) &= xe^{-x} \\ f'(x) &= e^{-x} - xe^{-x} \\ &= \frac{(1-x)}{e^x} < 0 \text{ when } x < 0 \end{aligned}$$

Taking the limit to find the lower bound is trivial using L'Hopital's Rule:

$$\lim_{n\to\infty} \frac{n}{e^n} \overset{\text{H}}{=} \lim_{n\to\infty} \frac{1}{e^n} = 0$$

The upper limit then is $a_1 = {}^1\!/_e$ as we've shown by the function theorem and the derivative that the sequence is always decreasing from that value.

11.1.8 (79, 80b, 81, 82) More recursive sequences

79. Let $a_n = \{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}...\}$; find the limit of the sequence.

Like (70b), this exercise asks us to find the limit when the sequence contains itself in each proceeding term. First define it more clearly:

$$\begin{aligned} a_1 &= \sqrt{2} \\ a_{n+1} &= \sqrt{2a_n} \end{aligned}$$

Again we consider that the limit of the sequence does not change depending on where we begin, so

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$$

which we can rewrite from the definition as

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n} = L$$

and using properties of limits, and the relationship $\lim_{n\to\infty} a_n = L$, we have

$$\sqrt{2 \lim_{n\to\infty} a_n} = \sqrt{2L} = L$$

$$\begin{aligned} 2L &= L^2 \\ 2 &= L \end{aligned}$$

so the limit of the sequence is $\lim_{n\to\infty} a_n = 2$.

---

81. Show that the sequence defined by $a_1 = 1$, $a_{n+1} = 3 - \frac{1}{a_n}$ is increasing and that $a_n < 3$ for all n. Deduce that $\{a_n\}$ is convergent and find its limit.

Assume that $\{a_n\}$ diverges and so its limit does not exist or equals infinity. Then taking its limit, we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$$

$$L = \lim_{n\to\infty} 3 - \frac{1}{a_n} = 3 - \frac{1}{\lim_{n\to\infty} a_n} = 3 - \frac{1}{L}$$

$$L + \frac{1}{L} = 3$$

$$L^2 + 1 = 3L$$

$$L^2 - 3L + 1 = 0$$

$$L = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 1 \times 1}}{2 \times 1}$$

$$\lim_{n\to\infty} a_n = L = \frac{3 \pm \sqrt{5}}{2}$$

The limit $\lim_{n\to\infty} a_n$ does exist and does not equal infinity, so $\{a_n\}$ does not diverge.

11.2 Series

Exercises to practice

1–56, 64, 67, 68

☐☐☐ **11.2.1 (1, 2) Define series; contrast with sequences**

☐☐☐ **11.2.2 (3, 4) Evaluate *partial sums* given a general formula S_n**

☐☐☐ **11.2.3 (5–8) Evaluate partial sums given the sequence of terms a_n**

☐☐☐ **11.2.4 (9–14) Analyze convergence of series graphically from partial sums**

☐☐☐ **11.2.5 (15) The relationship between a series and the sequence of its terms**

☐☐☐ **11.2.6 (16) Understand summation notation**

16 (a) Explain the difference between $\sum_{i=1}^{n} a_i$ and $\sum_{j=1}^{n} a_j$.

Both of these symbols represent the sum of entries in a sequence $\{a_n\}$. The first indicates summing the i^{th} entry as i goes from 1 to n, where i is an integer. The second indicates summing the j^{th} entry as j goes from 1 to n, where j is an integer. Assuming there is only one sequence called a, $a_i = a_j$ when $i = j$, so $\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j$.

Questions to ask

(b) Explain the difference between $\sum_{i=1}^{n} a_i$ and $\sum_{i=1}^{n} a_j$. The first is again the sum of entries $\{a_i\} = a_1 + a_2 + ... + a_n$, but the second is the sum of entries a_j where j is not explicitly defined in this question. If j is defined and a_j is an entry in $\{a_n\}$ then $\sum_{i=1}^{n} a_j = n \times a_j$.

☐☐☐ **11.2.7 (17–26) Analyze and evaluate *geometric series***

☐☐☐ **11.2.8 (27–42) Use partial sums, *Divergence Test*, *Geometric Series Test***

☐☐☐ **11.2.9 (43–48) Evaluate series by *telescoping* partial sums**

☐☐☐ **11.2.10 (49) Understand why $0.\bar{9} = 1$**

49. Let $x = 0.\bar{9}$ and consider the following:

(a) Do you think that $x < 1$ or $x = 1$? I generally believed that x is the largest value less than 1 that is not equal to 1. I have argued that $x = 1 - \epsilon$

where $\epsilon < N$ for any N such that $N > 0$.

(b) Sum a geometric series to find the value of x.

$$x = \sum_{n=1}^{\infty} 9\left(\frac{1}{10}\right)^n = 0.9 + 0.09 + 0.009 + ... = 0.99\overline{9}$$

x is defined by a geometric sequence with $r = \frac{1}{10}$ and $a = a_1 = 0.9$, so by the *Geometric Series Test Theorem* we can say that it converges ($|r| < 1$) and that it's value is

$$x = \frac{a}{1-r} = \frac{0.9}{1-\frac{1}{10}} = \frac{0.9}{0.9} = 1$$

By adding infinitely many terms of $\frac{9}{10^n}$ we do in fact get a sum of 1, but we see by the *Convergent Series Theorem* that the terms are ultimately infinitely small

$$\lim_{n\to\infty} \frac{9}{10^n} = \frac{9}{\infty} = 0$$

and so my original supposition that $x = 1 - \epsilon$ where $\epsilon < N$ for any $N > 0$ is only true in finite terms; in an infinite sum the limit of ϵ is 0 and $x = 1$.

(c) How many decimal representations does the number 1 have? One decimal representation ($0.99\overline{9}$) has been described in (b), and another is obviously

$$1 + \sum_{n=1}^{\infty} 0^n = 1.00\overline{0}$$

But I suspect that the geometric series producing $0.99\overline{9}$ has a symmetry where $1 = 1 + \epsilon$ as $\epsilon \to 0^+$.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_n = 1.000...1$$

(d) Which numbers have more than one decimal representation? It seems that we could repeat the process in (b) for any value of x:

$$x = \sum_{n=1}^{\infty} 80\left(\frac{1}{11}\right)^n = \frac{80}{11} + \frac{80}{121} + \frac{80}{1331} + ... = 7.99\overline{9}$$

$$x = \frac{{}^{80}\!/_{11}}{1-\frac{1}{11}} = \frac{{}^{80}\!/_{11}}{{}^{10}\!/_{11}} = \frac{80}{11} \times \frac{11}{10} = \frac{80}{10} = 8$$

This seems to work for $x \in \mathbb{R}$:

$$\sum_{n=1}^{\infty} 10 \times x \times \left(\frac{1}{11}\right)^n = \frac{10 \times x}{11} + \frac{10 \times x}{121} + ... = x - 1 + 0.99\overline{9}$$

$$\frac{a}{1-r} = \frac{\frac{10\times x}{11}}{1-\frac{1}{11}} = \frac{10\times x}{11} \times \frac{11}{10} = x$$

☐☐☐ **11.2.11 (50) Evaluate a series of recursive terms**

☐☐☐ **11.2.12 (51–56) Express repeating decimals as ratios of integers (see exercise 49)**

☐☐☐ **11.2.13 (64) Show that not all series whose terms go to zero will converge**

☐☐☐ **11.2.14 (67, 68) Derive a series from a formula for its partial sums**

11.3 The Integral Test and Estimates of Sums

Exercises to practice

1–26, 36–41

☐☐☐ **11.3.1 (1, 2) Understand the series/integral relationship graphically**

☐☐☐ **11.3.2 (3–8) Apply the *Integral Test* to determine convergence**

☐☐☐ **11.3.3 (9–26) Use partial sums, *DT, GST, IT* to determine convergence**

☐☐☐ **11.3.4 (36, 37) Approximate a series using a partial sum and estimate error with the *Remainder Estimate***

36. (a) Find the partial sum S_{10} of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Numerically estimated on a TI-84:

$$S_{10} = \sum_{n=1}^{\infty} \frac{1}{n^4} \approx 1.08203$$

Estimate the error in the approximation.

Questions to ask

$$R_{10} \leq \int_{10}^{\infty} \frac{dx}{x^4} = \left[-\frac{1}{3x^3}\right]_{10}^{\infty} = \lim_{t\to\infty}\left(-\frac{1}{3t^3}\right) - \left(-\frac{1}{3\times 10^3}\right) = 0 + \frac{1}{3000} = 0.000\overline{33}$$

(b) Use the double-ended remainder formula to get a better approximation. Equation $\boxed{3}$ in this section of the book uses both the over- and under-estimates of the series to give upper and lower accuracy bounds:

$$S_n + \int_{n+1}^{\infty} f(x)dx \leq s \leq S_n + \int_{n}^{\infty} f(x)dx$$

For our series,

$$S_{10} + \int_{11}^{\infty} \frac{dx}{x^4} \leq \sum_{n=1}^{\infty} \frac{1}{n^4} \leq S_{10} + \int_{10}^{\infty} \frac{dx}{x^4}$$

We found the antiderivative $\int \frac{dx}{x^4}$ and computed $\int_{10}^{\infty} \frac{dx}{x^4}$ in part (a); this is the upper bound of accuracy—that is, our estimate will be at worst *over*-estimated by this amount. Compute the antiderivative from $[11, \infty)$ to get the lower bound of accuracy—this will be the magnitude of our worst *under*-estimate:

$$\int_{11}^{\infty} \frac{dx}{x^4} = \left[-\frac{1}{3x^3}\right]_{11}^{\infty} = \lim_{t\to\infty}\left(-\frac{1}{3t^3}\right) - \left(-\frac{1}{3\times 11^3}\right) = 0 + \frac{1}{3993} \approx 0.00025$$

Taking the error bounds with our estimated sum,

$$1.08203 + 0.00025 \le s \le 1.08203 + 0.000\overline{33}$$
$$1.08228 \le \sum_{n=1}^{\infty} \frac{1}{n^4} \le 1.08236$$

(c) The exact value of this series is $\frac{\pi^4}{90}$; how does your estimate compare? To five decimals of precision, $\frac{\pi^4}{90} \approx 1.08232$, which lies directly between our two accuracy bounds. Our estimate is correct in the first three places, an error of 290 parts per million.

(d) Find a value of n so that $S_n$is within 0.00001 of the sum

---

11.3.5 (38, 39) Approximate a series to a specified precision

38. Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ correct to three decimal places.

Three decimals of precision means we need to pick an n such that $R_n \le 0.0001$. We can solve the inequality

$$R_n \le \int_n^{\infty} \frac{1}{x^5} dx = \left[-\frac{1}{4x^4}\right]_n^{\infty} = \lim_{t\to\infty}\left(-\frac{1}{4t^4}\right) - \left(-\frac{1}{4n^4}\right) = \frac{1}{4n^4}$$

$$R_n \le \frac{1}{4n^4} \le 0.0001$$
$$1 \le 0.0001 \times 4n^4$$
$$n \ge \sqrt[4]{2500} \approx 7.07$$

So we need at least 8 terms. Using the double-ended remainder in (36b), we get upper and lower bounds for the value of the sum

$$S_8 + \int_9^{\infty} \frac{1}{x^5} dx \le s \le S_8 + \int_8^{\infty} \frac{1}{x^5} dx$$

substituting the antiderivative from above,

$$S_8 + \frac{1}{4\times 9^4} \le s \le S_8 + \frac{1}{4\times 8^4}$$

then we compute S_8 numerically and complete the substitution

$$S_8 \approx 1.03688$$

$$1.03691 \leq s \leq 1.03694$$

taking s as the midpoint of the error interval, we have $s \approx 1.036925$ with zero error in the third decimal place.

□□□ 11.3.6 (40) Systematically increase precision of an approximation

□□□ 11.3.7 (41) Understand the mechanics of precision in partial sums

11.4 The Comparison Tests

Exercises to practice

1–37

☐☐☐ **11.4.1 (1, 2) Grasp the logic of comparison and convergence/divergence**

☐☐☐ **11.4.2 (3–32) Use the *Comparison Test* to determine convergence; practice choosing comparable series**

☐☐☐ **11.4.3 (33–36) Practice approximating series and estimating error**

☐☐☐ **11.4.4 (37) Practice understanding of repeating decimals as series**

Questions to ask

11.5 Alternating Series

Exercises to practice

1–30

☐☐☐ **11.5.1 (1) Define alternating series**

☐☐☐ **11.5.2 (2–20) Apply the *Alternating Series Test* to determine convergence**

☐☐☐ **11.5.3 (21–22) Apply the *Alternating Series Estimation Theorem***

☐☐☐ **11.5.4 (23–26) Approximate alternating series to a desired precision**

☐☐☐ **11.5.5 (27–30) Practice approximating alternating series**

Questions to ask

11.6 Absolute Convergence and the Ratio and Root Tests

Exercises to practice

1–30

☐☐☐ **11.6.1** **(1) Interpret the results of the *Ratio Test***

☐☐☐ **11.6.2** **(2–30) Use the *Root* and *Ratio* tests to determine *Absolute Convergence*; practice all other skills to determine *Conditional Convergence***

Questions to ask

11.7 Strategy for Testing Series

Exercises to practice

1–38

11.7.1 (1–38) Determine series convergence using all available tools

These exercises ask us to analyze each series and identify the best theorem for testing the series for convergence. One series may require applying more than one theorem from Chapter 11; for example, if we claim $\sum a_n \leq \frac{1}{n^2}$, we're invoking the *Direct Comparison Test*—but we would also need to refer to the *P-Series* theorem to show why comparing with $\sum \frac{1}{n^2}$ proves convergence.

This process requires us to notice features of a series that leave it vulnerable to attack with the different tests. For example, a series defined with $(-1)^n$ is an obvious candidate for the *alternating series test*; while a series with powers of n in the numerator and denominator should succumb to the *root test*.

6. $\sum \frac{1}{2n+1}$

direct comparison with the *divergent harmonic series* fails because this series is smaller; comparing with a divergent series only proves divergence if the compared series is *larger* than the divergent one (section 11.4, exercises 1–2).

Instead we can use *limit comparison* to establish that there is a finite nonzero ratio between the two series at their limit, and so if one diverges the other must as well:

$$\lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{n}} = \frac{n}{2n+1} \overset{\text{H}}{=} \frac{1}{2}$$

Questions to ask

The ratio of the two series at the limit as $n \to \infty$ is $1/2$, so the series $\sum \frac{1}{2n+1}$ is divergent by limit comparison with the divergent harmonic series.

11.8 Power Series

Exercises to practice

1–30, 33

11.8.1 (1–2) Define a *power series* and identify its characteristics

1. What is a power series?

A power series is essentially an infinite polynomial—it's a polynomial of degree n, but $n \to \infty$

$$y = a + bx + cx^2 + \cdots$$

We're equipped now to represent this idea as a series, and we call it a *power series*

$$\sum_{n=0}^{\infty} C_n(x-a)^n = C_0 + C_1(x-a)^1 + C_2(x-a)^2 + \cdots$$

Notice this series has a constant term, a linear term, a quadratic term, etc. We have a sequence of coefficients $\{C_n\}$ so that each term can be weighted separately; some terms can even be eliminated when $C_i = 0$. Whatever the coefficient, the series includes every degree of polynomial.

Notice also that the series is *phase shifted* by an amount a. Like a trigonometric phase shift, a positive value of a shifts the series to the right because it begins sampling at an earlier point in the input space. A power series of the form $\sum_{n=0}^{\infty} C_n(x+a)^n$ should be considered as having a negative value for a and therefore being left-shifted. **The value of a is called the *center* of the power series**, and the above series is called *a power series centered about a.*

Questions to ask

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2. (a) What is the radius of convergence of a power series? How do you find it?

Because the value of x in the series above is variable, there may be values of x for which the series converges and values for which it diverges. As x values grow larger or smaller we consider the difference $|x-a|$, or the *radius* from the *center* of the series. The greatest difference within which the series still converges is called the *radius of convergence*. That is, if the radius of convergence is R then the series will converge for any x such that $|x-a| < R$. Whether a series converges at $|x-a| = R$ must be determined separately for $x = a - R$ and $x = a + R$.

(b) What is the interval of convergence of a power series? How do you find it? We can express the above statements about the *radius of convergence* as an interval of x values for which the series converges. That interval, called the *interval of convergence*, extends in the positive and negative directions from the *center* by magnitude R. Again, whether the series converges at $x = a - R$ and $x = a+R$ must be determined separately; and so there are four possible intervals of convergence that differ by whether they include or exclude the endpoints:

$$(a - R, a + R)$$
$$(a - R, a + R]$$
$$[a - R, a + R)$$
$$[a - R, a + R]$$

In addition to these there are two possible intervals for special cases of the radius:

$$\{a\}, \text{ when } R = 0$$
$$(-\infty, \infty), \text{ when } R = \infty$$

The first special case demonstrates that the interval of convergence always contains the center of the series. When the radius is infinite the series converges for all real numbers.

Consider the geometric series $\sum_{n=1}^{\infty} x^n$, where $r = x$. By the *Geometric Series Test* this series converges for all x such that $|x| < 1$

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}, \quad -1 < x < 1$$

Put another way, the series converges for all x on the *interval of convergence* $(-1, 1)$. Notice that this interval does not include its endpoints, because a geometric series diverges when $|r| = 1$.

☐☐☐ **11.8.2 (3–28) Find the radius and interval of convergence of the series**

☐☐☐ **11.8.3 (29) Understand the relationship between the radius and behavior of a power series**

☐☐☐ **11.8.4 (30) Compare the convergence behavior of different power series**

11.9 Representing Functions as Power Series

Exercises to practice

1–32

This section brings together all of the series-testing skills learned so far in Chapter 11. Approximations will come next, but for now we will be tasked to

1. Identify to which of a few elementary functions a given function is equivalent
2. Manipulate that function using algebra and calculus until it is written in the form of the elementary function
3. Represent that manipulated function as a series that is equivalent to the elementary form
4. Test that series representation for convergence under a few conditions

Steps 1–3 are new to this section, while step 4 is a summary of everything we've covered so far. **The most important skill** we develop in this section will be identifying the elementary forms, manipulating the given functions to fit those forms, and writing the equivalent power series. Each of these forms and its equivalent series should be memorized from a table. Additionally, each of these elementary series will be often tested for convergence and those tests should be practiced consistently.

☐☐☐

11.9.1 (1–2) Understand how calculus operators affect the radius of convergence of a power series

Questions to ask

1. If the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ is 10, what is the radius of convergence of the series $\sum_{n=1}^{\infty} n c_n x^{n-1}$? Why?

Theorem [2] in Stewart (p. 748) states that if a function is represented by a power series with radius of convergence $R > 0$, then that function is continuous and differentiable on the interval of convergence $(a - R, a + R)$. Theorem [2] also states that differentiating or integration such a function does not change its radius of convergence, though the endpoint behavior may change. Considering the series in the question, we can see that

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$\frac{df}{dx} = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

The function f represented by the first power series is continuous and differentiable by $\boxed{2}$, and we can see that the second power series represents the derivative $\frac{df}{dx}$. Notice that the series representation of the derivative begins at $n = 1$; we choose n such that the first term of the derivative series is the derivative of the first term of the function series. Likewise for integrating a power series, we choose n such that the first term of the integral series is the integral of the first term of the function series.

By theorem $\boxed{2}$ then we can say that the radius of convergence of these two series is the same; however the convergence behavior at the endpoints may differ, and so each endpoint must be tested for convergence in the new series.

To summarize, given the following power series representations of functions:

$$\begin{aligned} f(x) &= \sum a_n \\ \frac{dx}{dx} &= \sum b_n \\ \int f(x)dx &= \sum c_n \end{aligned}$$

Each of these series has the same radius of convergence, though the interval may differ in its inclusion of either endpoint.

Consider that $\frac{d}{dx}$ and $\int dx$ are linear operators; what property do you think produces their relationship with the radius of convergence?

---

2. Suppose you know that the series $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x| < 2$. What can you say about the following series? Why?

$$\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$$

Rewriting we have

$$\sum_{n=0}^{\infty} b_n \frac{x^{n+1}}{n+1}$$

Now consider the following

$$\begin{aligned} f(x) &= bx^n \\ \int f(x) &= \int bx^n = b \int x^n \\ &= b\frac{x^{n+1}}{n+1} \end{aligned}$$

Comparing $f(x)$ and its integral with the series above we can plainly see that the series is the integral of the known convergent series, and that n and b_n are integrated as constants. Thus we say that

$$\int \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1} = I(x)$$

and that the series $I(x)$ converges for $|x| < 2$ ($\boxed{2}$ in Stewart p. 748). Whether $I(x)$ converges for $x = \pm 2$ can't be determined without testing those endpoints individually, which we do

$$I(-2) = \sum_{n=0}^{\infty} \frac{b_n}{n+1} (-2)^{n+1}$$

by $\boxed{\text{AST}}$,

(1) $\dfrac{2^{n+2} b_n}{n+1} > \dfrac{2^{n+1} b_{n+1}}{n+2}$ when $b_n \geq b_{n+1}$

~~(2)~~ $\lim_{n\to\infty} \dfrac{2^{n+1} b_n}{n+1} \overset{\text{H}}{=} \lim_{n\to\infty} \ln(2) b_n 2^{n+1} = \infty$

diverges for $x = -2$ by $\boxed{\text{DT}}$

$$I(2) = \sum_{n=0}^{\infty} \frac{2b_n}{n+1}$$

diverges for $x = 2$ when $b_n \geq 1$ by $\boxed{\text{LCOMP}}$ with harmonic series

so the interval of convergence is $(-2, 2)$.

□□□ **11.9.2 (3–10) Find a power series representation for the function using algebraic manipulation; determine the interval of convergence of that series**

Hint: consider how each function relates to the Geometric Series Theorem ($\boxed{4}$ in Stewart p. 706)

□□□ **11.9.3 (11–12) Represent the function as the sum of two power series and find the interval of convergence**

□□□ **11.9.4 (13–14) Differentiate the function to find a power series representation for it and determine the radius of convergence**

☐☐☐ **11.9.5 (15–20) Transform the function by algebra and/or calculus to find a power series representation**

☐☐☐ **11.9.6 (21–24) Find a power series representation of the function; graphically compare the behavior of the function with the behavior of partial sums of the series**

☐☐☐ **11.9.7 (25–28) Applications: use a power series representation to evaluate the indefinite integral**

☐☐☐ **11.9.8 (29–30) Applications: use a power series representation to approximate the definite integral**

11.10 Taylor and Maclaurin Series

5–10, 13–22, 25–36, 39–70

Exercises to practice

A taylor series is a general form of a series representation of a function; that is, instead of finding a known function with a known series representation, we can construct series representations of arbitrary functions. The elementary series representations we learned in section 9 are the products of applying the *Taylor process* to functions of those elementary forms. We use the elementary representations as shortcuts, as many functions will fit those forms without needing to do a lengthy construction of the equivalent series. Being creative with transformation by algebra or calculus will allow us to identify the elementary forms and arrive at a solution while skipping many steps in the series construction.

In this section we learn what we can do when a function does not fit any elementary form (and how the series for those forms can be constructed). The major constraint on the Taylor process is that the function must have a power series representation. This constraint comes naturally from an understanding:

> Power series represent an infinite-dimension vector space containing all real-valued continuously differentiable functions. This is the polynomial space $P[x]$.

It's evident then that no power series can represent a function that isn't continuously differentiable, because there is no vector in the polynomial space that is equivalent to such a function. A more intuitive notion further supports this constraint:

> There is no polynomial that can pass through a corner because polynomials are smooth and continuous and a corner is not smooth.

Questions to ask

☐☐☐ **11.10.1 (5–10) Find the Maclaurin series for the given function; find the the radius of convergence**

☐☐☐ **11.10.2 (13–20) Find the Taylor series for the given function and center; find the radius of convergence**

☐☐☐ **11.10.3 (20–22) Prove that the series found in a previous exercise represents the given function**

☐☐☐ **11.10.4 (25–28) Use the *binomial series* to expand the function as a power series; find the radius of convergence**

☐☐☐ **11.10.5 (29–36) Use a Maclaurin series from the reference table to find the Maclaurin series for the given function**

☐☐☐ **11.10.6** **(39–42) Find the Maclaurin series for the given function, and its radius of convergence. Compare graphically the function with its first few Taylor polynomials**

☐☐☐ **11.10.7** **(43–44) Use the Maclaurin series for the given function to approximate the function's value at the given input**

☐☐☐ **11.10.8** **(45) Expand the given function as a binomial series; use the result to find the Maclaurin series for a related function (see section §7.3 *Trig Substitution* on page 15)**

☐☐☐ **11.10.9** **(46) Expand the given function as a power series and use the series to approximate the function's value at the given input**

☐☐☐ **11.10.10** **(47–50) Evaluate the indefinite integral as an infinite series**

☐☐☐ **11.10.11** **(51–54) Use series to approximate the definite integral to the indicated accuracy**

☐☐☐ **11.10.12** **(55–58) Use series to evaluate the limit**

☐☐☐ **11.10.13** **(59–62) Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for each function**

☐☐☐ **11.10.14** **(63–70) Evaluate the given series (find its equivalent function)**

11.11 Applications of Taylor Polynomials

Exercises to practice

3–10, 13–26ab, 27–29

☐☐☐ **11.11.1** **(3–10) Find the degree 3 Taylor polynomial $T_3(x)$ for the function f centered at a; compare the approximation to the function graphically**

☐☐☐ **11.11.2** **(13–22) (a) Approximate f by $T_n(x)$ at center a; (b) Use Taylor's Inequality to estimate the accuracy of the approximation on the given interval**

☐☐☐ **11.11.3** **(23–24) Use the series from section 11.11.1 to approximate trigonometric functions**

☐☐☐ **11.11.4** **(25–26) Determine how many terms of the Maclaurin series for the given function are needed to approximate its value to the specified accuracy**

☐☐☐ **11.11.5** **(27–29) Use *Alternating Series Estimation Theorem* or Taylor's Inequality to estimate the interval over which the given approximation is within the specified accuracy**

Questions to ask