MAT 231

Chapter 11 Part 1 - Exam Solutions

Calculus: Early Transcendentals, 7th Edition, James Stewart

Matt Barnard

1 Determine if the sequence converges or diverges; if it converges find the value.

A) $\left\{\frac{3+5n^2}{n+n^2}\right\}$

We can take the limit directly and find that it exists with a finite value, therefore the sequence converges. The limit of the sequence as written has the indeterminate form ∞/∞, so we use L'Hopital's Rule

$$\lim_{n\to\infty}\frac{3+5n^2}{n+n^2} \overset{\text{H}}{=} \lim_{n\to\infty}\frac{10n}{1+2n} \overset{\text{H}}{=} \lim_{n\to\infty}\frac{10}{2} = 5$$

or we can use algebra to rewrite the indeterminate form as a determinate one

$$\lim_{n\to\infty}\frac{3+5n^2}{n+n^2}\times\frac{1/n^2}{1/n^2} = \lim_{n\to\infty}\frac{3/n+5}{1/n+1} = \frac{0+5}{0+1} = 5$$

and so the sequence converges to 5.

I had originally tried to show convergence by comparing the sequence with a convergent one

$$0 \le \lim_{n\to\infty}\frac{3+5n^2}{n+n^2} \le \lim_{n\to\infty}\frac{5n^2}{n^2} = 5$$

which can be proved by comparing the limits after applying L'Hopital's Rule, as both are of indeterminate form ∞/∞

$$\lim_{n\to\infty}\frac{10n}{1+2n} < \lim_{n\to\infty}\frac{10n}{2n}$$

which is true when $n > 0$. The comparison test for convergence does not give the value, however, and so is an unnecessary step in answering this question.

B) $\left\{\frac{\ln(n)}{\ln(2n)}\right\}$

This sequence is of indeterminate form ∞/∞, so we again use L'Hopital's Rule

$$\lim_{n\to\infty}\frac{\ln(n)}{\ln(2n)} \overset{\text{H}}{=} \lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{2}{2n}} = \lim_{n\to\infty}\frac{1}{n}\times\frac{2n}{2} = 1$$

---

2 Assume the sequence $\{a_n\}$, where $a_1 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$, is convergent. Determine $\lim_{n\to\infty} a_n$.

This is a recursive sequence that expands to

$$\left\{1, \frac{1}{1+1}, \frac{1}{1+\frac{1}{1+1}}, \cdots\right\}$$

Taking the limit of this sequence depends on the relationship $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = L$ (see Appendix A on page 9 for a proof of this equality). Taking L as the limit of a_n and the definition of a_{n+1}, we have

$$\begin{aligned}\lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty}\frac{1}{1+a_n}\\ L &= \frac{1}{1+L}\\ L + L^2 &= 1\\ L^2 + L - 1 &= 0\end{aligned}$$

which does not factor, so we use the quadratic formula with $a = 1, b = 1, c = -1$

$$L = \frac{-1\pm\sqrt{1^2-4(1)(-1)}}{2} = \frac{-1\pm\sqrt{5}}{2}$$

Now, the sequence is strictly positive when $n > 0$ so the negative solution for L cannot be the limit of the sequence. Simplifying, we finally have the limit

$$\lim_{n\to\infty} a_n = \frac{\sqrt{5}-1}{2}$$

which is equal to $\varphi - 1$ where φ is the *golden ratio* (the positive solution to $\varphi^2 - \varphi - 1 = 0$).

---

3 The series below are convergent. Determine the value of each series.

A) $\displaystyle\sum_{n=1}^{\infty}\left(e^{\frac{1}{n}} - e^{\frac{1}{n+1}}\right)$

We can immediately take notice that when the term at n is added the subsequent term is subtracted; at $n+1$ the previously subtracted term is now the added term and thus is cancelled. If we write out a few terms of the series (where a_n is their sequence) we see

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \sum_{i=1}^{n} a_i$$

$$S_n = (e^1 - e^{1/2}) + (e^{1/2} - e^{1/3}) + (e^{1/3} - e^{1/4}) + \cdots + (e^{1/n-1} - e^{1/n}) + (e^{1/n} - e^{1/n+1})$$

Regrouping and simplifying, we have

$$S_n = e^1 + \cancel{(-e^{1/2} + e^{1/2})} + \cancel{(-e^{1/3} + e^{1/3})} + \cancel{(-e^{1/4} + \cdots + e^{1/n-1})} + \cancel{(-e^{1/n} + e^{1/n})} - e^{1/n+1})$$

$$S_n = e - e^{\frac{1}{n+1}}$$

Taking the limit of the partial sum S_n as $n \to \infty$ we find the value of the series

$$\sum_{n=1}^{\infty}\left(e^{\frac{1}{n}} - e^{\frac{1}{n+1}}\right) = \lim_{n\to\infty} e - e^{\frac{1}{n+1}} = e - e^0 = e - 1$$

B) $10 - 2 + 0.4 - 0.08 + \cdots$

First we should notice that the terms are alternating in sign so we should expect to find a $(-1)^n$ term in the series formula. Beyond that we have few immediate options to analyze these terms, the most obvious being a common rational relationship:

$$\frac{a_2}{a_1} = \frac{-2}{10} = -\frac{1}{5}$$

$$\frac{a_3}{a_2} = \frac{0.4}{-2} = -\frac{1}{5}$$

$$\frac{a_4}{a_3} = \frac{-0.08}{0.4} = -\frac{1}{5}$$

So we see that this is a *geometric series* of the form $\sum_{n=1}^{\infty} ar^{n-1}$ with $r = -\frac{1}{5}$, and it is convergent by the *geometric series test* as $|r| < 1$ ($\boxed{4}$ in Stewart p. 706).

If we were to factor $\left(-\frac{1}{5}\right)^{n-1}$ into $(-1)^{n-1}\left(\frac{1}{5}\right)^{n}$ we see the $(-1)^n$ that we noticed previously. The first term $a_1 = 10$, so we solve for a

$$a\left(-\frac{1}{5}\right)^{0} = 10$$
$$a = 10$$

completely specifying the series. Using the equation for the value of a geometric series ($\boxed{4}$ again in Stewart), we solve

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$
$$\sum_{n=1}^{\infty} 10 \times \left(-\frac{1}{5}\right)^{n-1} = \frac{10}{1-(-\frac{1}{5})} = \frac{10}{6/5} = \frac{25}{3}$$

A mistake I made on the exam is that I decided to factor the $(-1)^n$ early as I stated above, so when I searched for the ratio between terms I used their absolute value, expecting to reinsert the alternating signs when writing the series formula. As a result I found $r = \frac{1}{5}$, which I justified by writing it as $|r| = \frac{1}{5} < 1$, mixing the definition of r with the convergence test. When solving the value of the series I abandoned the factor of $(-1)^n$ and wrote the value as $\frac{10}{1-1/5} = \frac{25}{2}$.

---

4 Determine if the series are convergent or divergent.

A) $$\sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1}$$

This series stands out as "an algebraic function of n (involving roots of polynomials)" (Stewart p. 739) so we try to coerce it to compare with a *p-series.*

At first glance the largest degree is in the denominator so we assume the terms have decreasing behavior. The $\sqrt{n}$ term in the numerator carries insignificant weight against the linear n term in the denominator, as we see the behavior of that ratio is strictly decreasing when $n > 0$:

$$\frac{d}{dx}\frac{\sqrt{n}}{n} = \frac{\frac{n}{2\sqrt{n}} - \sqrt{n}}{n^2} = -\frac{1}{2n^{3/2}}$$

So we conclude that the terms of this series are likely decreasing—and likely their limit is zero so we skip the *divergence test.*

We'd like to find that it's smaller than some convergent p-series; the $\sqrt{n}$ term in the numerator will make direct comparison difficult, though, because the known-convergent p-series form $\frac{1}{n^p}$ has a constant numerator. Direct comparison demands that our series is smaller than the comparator, which won't be true when the numerator is larger than 1. Still we claim this series converges, and we compare its behavior with that of a convergent p-series at the limit; if there is a finite nonzero ratio between the limits of their terms then they share convergence behavior and our claim is supported.

To choose a comparable series we simply discard terms or factors that have little impact on the end behavior

$$\frac{\sqrt{n+2}}{2n^2+n+1} \Rightarrow \frac{\sqrt{n}}{2n^2+n+1} \Rightarrow \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = \sum_{n=1}^{\infty} \frac{1}{n^p}, \;\; p = \frac{3}{2} > 1 \quad \text{(converges)}$$

Comparing the ratio of the terms of these series at the limit, we see that

Hint: get n in denominators to take terms to zero

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{\sqrt{n+2}}{2n^2+n+1}}{\frac{1}{n^{3/2}}} &= \lim_{n\to\infty} \frac{\frac{\cancel{n^2}}{\sqrt{n}}\sqrt{n+2}}{\cancel{n^2}\left(2+\frac{1}{n}+\frac{1}{n^2}\right)} \\ &= \lim_{n\to\infty} \frac{\sqrt{1+{}^2\!/\!_n}}{2+{}^1\!/\!_n+{}^1\!/\!_{n^2}} \\ &= \frac{\sqrt{1+0}}{2+0+0} = \frac{1}{2} \end{aligned}$$

the ratio is finite and nonzero, so

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{2n^2+n+1} \text{ converges by limit comparison with } \sum_{n=1}^{\infty} \frac{1}{n^p}, \;\; p = \frac{3}{2}$$

B) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$

This series gives us a great example of the process of finding similar series for comparison; as it turns out we also see a demonstration of why performing the *divergence test* as a first approach can be valuable. We're reminded to practice using algebra to find forms where infinitely increasing terms are divisors (see Problem 4[A]).

Taking the limit of the terms (which ought to go to 0 if the series is to converge, $\boxed{7}$ in Stewart p. 709), we have

$$\lim_{n\to\infty} \frac{n+1}{n\sqrt{n}} \times \frac{{}^1\!/\!_n}{{}^1\!/\!_n} = \lim_{n\to\infty} \frac{1+{}^1\!/\!_n}{\sqrt{n}} = 0$$

We might be tempted to rush to conclude that the divergence test was inconclusive in this case. Looking more closely, we actually exposed a relationship that will be critical to determining convergence. Clearly $\lim_{n\to\infty} {}^{1}\!/\!_{n} = 0$, so we are justified in saying

$$\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1+{}^{1}\!/\!_{n}}{\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1+0}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$
$$\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \quad \text{(diverges)}$$

so our series diverges by direct comparison with a divergent p-series ($p = {}^{1}\!/\!_{2} \leq 1$).

C) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$

Closely resembling the harmonic series, this series might diverge if the terms weren't alternating in sign (indeed if we hold the sign constant, limit comparison does show that divergence). The limit of these terms does not exist, so if the *alternating series test* can't save it then this series will be divergent by the divergence test.

It's clear at a glance that the series satisfies the two critera of a convergent alternating series (Stewart p. 727). Given the series as $\sum a_n$ and $b_n = |a_n|$, we see that

1. $b_n \geq b_{n+1}$ for all n
$$\frac{1}{2n+1} > \frac{1}{2(n+1)+1} = \frac{1}{2n+3}$$
2. $\lim_{n\to\infty} b_n = 0$
$$\lim_{n\to\infty} \frac{1}{2n+1} = 0 < 1$$

so the series does converge by the alternating series test.

D) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

A clearer case for the *root test* could not be made. The series converges if we find this limit to have a finite value $L < 1$ (Stewart p. 736):

$$L = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n^2+1}{2n^2+1}\right)^n\right|} = \lim_{n\to\infty} \left(\frac{n^2+1}{2n^2+1}\right)^{n/n} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1}$$

which can be evaluated algebraically,

$$\lim_{n\to\infty} \frac{n^2+1}{2n^2+1} \times \frac{1/n^2}{1/n^2} = \lim_{n\to\infty} \frac{1+1/n^2}{2+1/n^2} = \frac{1+0}{2+0} = \frac{1}{2} < 1 \quad \text{(converges)}$$

or by L'Hopital's Rule

$$\lim_{n\to\infty} \frac{n^2+1}{2n^2+1} \overset{\text{H}}{=} \lim_{n\to\infty} \frac{2n}{4n} = \lim_{n\to\infty} \frac{2}{4} = \frac{1}{2} < 1 \quad \text{(converges)}$$

---

5 Determine if the series are absolutely convergent, conditionally convergent, or divergent.

A) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{5n+1}$

This is virtually the same series seen in Problem 4(C) on the previous page, but we're now tasked to determine absolute or conditional convergence. Because proving absolute convergence also proves convergence we attempt that first. Only if we fail to prove absolute convergence will we resort to proving conditional convergence by the method identical to that in Problem 4(C).

Taking the absolute value of the series as $\sum |a_n|$, we test convergence by limit comparison with the divergent harmonic series with terms $b_n = 1/n$, as briefly mentioned in Problem 4(C):

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{5n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{5n+1} \times \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{1}{5+1/n} = \frac{1}{5}$$

The ratio of the series terms at the limit is finite and nonzero so the series is not absolutely convergent.

Falling back on the alternating series test, we expect already that the series converges conditionally from our experience with Problem 4(C). Where the series is $\sum a_n$ and $b_n = |a_n|$, we test the two conditions:

1. $b_n \geq b_{n+1}$ for all n

$$\frac{1}{5n+1} > \frac{1}{5(n+1)+1} = \frac{1}{5n+6}$$

2. $\lim_{n\to\infty} b_n = 0$

$$\lim_{n\to\infty} \frac{1}{5n+1} = 0 < 1$$

and the series does converge conditionally.

B) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$

There is little we can do with a factorial but expand it, and this factorial makes for a very neatly evaluated limit when expanded in the ratio test:

$$\lim_{n\to\infty} \left| \frac{\frac{(-3)^{n+1}}{(2(n+1)+1)!}}{\frac{(-3)^n}{(2n+1)!}} \right| = \lim_{n\to\infty} \left(\frac{3^{n+1}}{(2n+3)!} \times \frac{(2n+1)!}{3^n} \right)$$

Regrouping, we expand the factorial in the denominator and cancel factors

$$L = \lim_{n\to\infty} \left(\frac{\cancel{3^n}3}{\cancel{3^n}} \times \frac{\cancel{(2n+1)!}}{(2n+3)(2n+2)\cancel{(2n+1)!}} \right) = \lim_{n\to\infty} \frac{3}{(2n+3)(2n+2)} = 0$$

so the series is absolutely convergent by the ratio test with $L = 0 < 1$.

A Proof that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$

Assume that $\lim_{n\to\infty} a_n = L$ and that L exists with a finite value. Using the definition of a convergent sequence ($\boxed{2}$ in Stewart p. 692), we can say that for any $\varepsilon > 0$ there is some N such that if $n > N$ then $|a_n - L| < \varepsilon$. It follows that because $n + 1 > n > N$, it's also true that $|a_{n+1} - L| < \varepsilon$; so we see that if a_n converges to L then the subsequence a_{n+1} also converges to L. Additionally, there is an ε' and an M such that $n + 1 > M > n > N$ and $|a_{n+1} - L| < \varepsilon' < |a_n - L| < \varepsilon$.